Homework set # 2 for the course
"Opinion Dynamics on Social Networks"

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1. **Epidemic spreading with awareness.** Consider a variant of the network SIS epidemic model, called SIAS (Susceptible-Infected-Aware-Susceptible). In this model, when a susceptible gets in contact with infected it can become aware, instead of infected (think of an individual interacting with flat-earth believers: no everybody is willing to get convinced by such an interaction...). An aware individual can still be infected but with a lower rate. The possible transitions and the associated parameters used in the model are shown in Fig. 1. The state variables are s_i = susceptible, p_i =

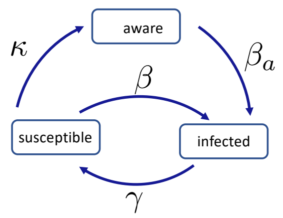

Figure 1: States transitions and parameters for SIAS epidemic model.

infected, q_i = aware, and the ODEs for each of the $i = 1, \ldots, n$ nodes are:

$$\begin{aligned} \dot{p}_i &= \beta(1 - p_i - q_i)\sum_j a_{ij}p_j + \beta_a q_i \sum_j a_{ij}p_j - \gamma p_i \\ \dot{q}_i &= \kappa(1 - p_i - q_i)\sum_j a_{ij}p_j - \beta_a q_i \sum_j a_{ij}p_j \end{aligned}$$

where as expected $s_i = 1 - p_i - q_i$ (hence no equation for s_i is needed). Notice that the parameters β, β_a, κ and γ are the same for all nodes. The fact that aware individuals have a lower chance of getting infected translates into $0 < \beta_a < \beta$. The aim of this exercise is to compare this model with a standard SIS model, like the one introduction in Section 7.4.2 of the Lecture Notes:

$$\dot{x}_i = \beta_i(1 - x_i)\sum_{j=1}^{n} a_{ij}x_j - \gamma_i x_i$$

where x_i = infected, and β and γ are the same as above.

(a) Show that when starting from identical initial conditions, $p_i(0) = x_i(0)$ for all i, then $p_i(t) \leq x_i(t)$ for all $t \geq 0$ (i.e., the aware state reduces the infection).

(b) show qualitatively (e.g. through simulations) that the behavior of the model is similar to that of the network SIS model, but that it has 2 thresholds $\tau_1 = 1$ and $\tau_2 > 1$. In particular you should show:

   i. if R_0 is the reproductive number from the SIS model, then $R_0 < \tau_1 = 1$ still implies convergence to the disease-free state (quick die-out: exponential convergence rate);

   ii. if $R_0 > \tau_1 = 1$ but $1 < R_0 < \tau_2$ (where τ_2 is a second threshold, $\tau_2 > 1$, very difficult to compute analytically) then while the SIS model converges to endemic epidemic, the SIAS model still converges to the disease-free state, even though with a much slower rate (slow die-out);

   iii. for $R_0 > \tau_2$ then also the SIAS model converges to an endemic epidemic state.

(c) In the endemic state, show that the higher is κ the higher is the fraction of aware, and also the lower is that of infected (you can show this numerically or analytically);

2. **A nonlinear consensus problem.** Consider the following nonlinear variant of a consensus process among n agents

$$\dot{x} = -D(x)Lx \tag{1}$$

where L is the Laplacian matrix of a connected undirected graph $\mathcal{G}(A)$, and $D(x)$ is a diagonal matrix which is a function of x. $D(x)$ can take one of the following 3 forms

$$D(x) = I - \mathrm{diag}(x)^2 \tag{2}$$
$$D(x) = I - \mathrm{diag}(x) \tag{3}$$
$$D(x) = \mathrm{diag}(x)^2 \tag{4}$$

(a) Show that for each choice of $D(x)$ the system (1) is forward invariant in $[-1, 1]^n$ (i.e., $x(0) \in [-1, 1]^n \Longrightarrow x(t) \in [-1, 1]^n \; \forall \, t > 0$).

(b) compute the equilibria of the system (1) in the 3 cases.

(c) Can you investigate the stability properties of these equilibria?

(d) Describe, through analysis or simulation, how the term $D(x)$ is altering the consensus behavior of the system in the three cases.