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Lecture 7

- Well-posedness
- Internal stability
- Youla Parameterization
- Coprime factorizations over $\mathcal{RH}_\infty$

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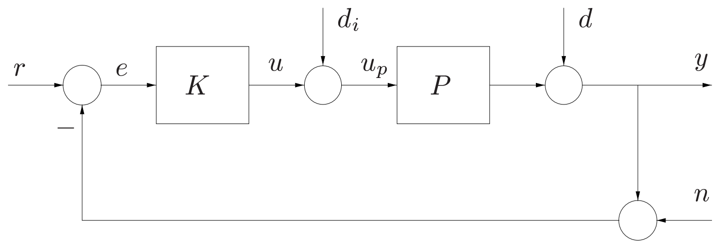
Standard Feedback Configuration



Assumption: In this lecture it is assumed that P, K are rational and proper.

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Well-Posedness

Example:

$$P = -\frac{s-1}{s+2}; \qquad K = 1$$

are both proper, but

$$u = \frac{s+2}{3}(r-n-d) - \frac{s-1}{3}d_i$$

Definition 5.1: A feedback system is said to be **well-posed** if all closed loop transfer function matrices are well-defined and proper.

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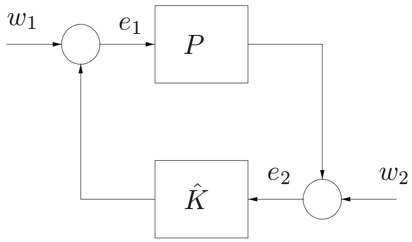
Internal Stability Analysis Diagram



The above diagram is equivalent to the standard feedback configuration if $\hat{K} = -K$ and

$$e_1 = u_p; \quad w_1 = d_i$$
$$e_2 = -e; \quad w_2 = n + d - r$$

Realization for the Closed Loop

Introduce a realization for the closed loop as

$$\dot{x} = Ax + Be_1, \qquad \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}e_2$$
$$e_2 = Cx + De_1 + w_2, \qquad e_1 = \hat{C}\hat{x} + \hat{D}e_2 + w_1$$

Introduce

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Realization for the Closed Loop continued

Then the closed loop satisfies

$$\begin{bmatrix} -sI + \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} \bar{x} \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} w$$

Lemma 5.1: Well-posedness is equivalent to invertibility of either of the following three matrices

$$\bar{D}; \quad I - \hat{D}D; \quad I - D\hat{D}$$

Remark: Notice that $D = P(\infty)$ and $\hat{D} = \hat{K}(\infty)$.

Proof

Eliminating $\bar{x}$ in

$$\begin{bmatrix} -sI + \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} \bar{x} \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} w$$

yields

$$w = \left(\bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}\right) e$$

So the closed loop system from w to e has well-defined and proper transfer functions if and only if $\bar{D}$ is invertible. Multiplying $\bar{D}$ with either

$$\begin{bmatrix} I & 0 \\ D & I \end{bmatrix}, \text{ or } \begin{bmatrix} I & \hat{D} \\ 0 & I \end{bmatrix}$$

from the left proves the other equivalences.

Internal Stability

In the remainder of the lecture well-posedness will be assumed.

Definition 5.2: The closed loop system is said to be **internally stable** if its realization (with w_1 and w_2 zero) is exponentially stable.

Lemma 5.2: The closed loop system is internally stable if and only if

$$\tilde{A} = \bar{A} + \bar{B}\bar{D}^{-1}\bar{C}$$

has all its eigenvalues in the open left half-plane.

Proof: When w_1 and w_2 are zero it holds that $\dot{\bar{x}} = \tilde{A}\bar{x}$.

Lemma A

If the closed loop system is internally stable, then (A, B) and $(\bar{A}, \bar{B})$ are stabilizable, and (C, A) and $(\bar{C}, \bar{A})$ are detectable.

Proof: Write

$$\bar{D}^{-1} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

Then

$$\tilde{A} - \lambda I = \begin{bmatrix} A + BD_{12}C - \lambda I & BD_{11}\hat{C} \\ \hat{B}D_{22}C & \hat{A} + \hat{B}D_{21}\hat{C} - \lambda I \end{bmatrix}$$

Assume that (A, B) is not stabilizable. Then there exist unstable λ_0 and $p^T \neq 0$ such that $p^T \begin{bmatrix} A - \lambda_0 I & B \end{bmatrix} = 0$. Hence $\begin{bmatrix} p^T & 0 \end{bmatrix} (\tilde{A} - \lambda_0 I) = 0$ implying that $\tilde{A}$ is not stable. Hence (A, B) is stabilizable....

Lemma 5.3

The closed loop system is internally stable if and only if (A, B) and $(\bar{A}, \bar{B})$ are stabilizable, (C, A) and $(\bar{C}, \bar{A})$ are detectable, and

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - \hat{K}P)^{-1} & \hat{K}(I - P\hat{K})^{-1} \\ P(I - \hat{K}P)^{-1} & (I - P\hat{K})^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I + \hat{K}(I - P\hat{K})^{-1} & \hat{K}(I - P\hat{K})^{-1} \\ (I - P\hat{K})^{-1}P & (I - P\hat{K})^{-1} \end{bmatrix} \in \mathcal{RH}_\infty$$

Proof: First notice that

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} = \bar{D}^{-1}\bar{C}(sI - \tilde{A})^{-1}\bar{B}\bar{D}^{-1} + \bar{D}^{-1}$$

Proof continued

If the closed loop system is internally stable, then by Lemma 5.2 $\tilde{A}$ has all its eigenvalues in the open left half-plane. Moreover, it is assumed that the closed loop is well-posed. Hence

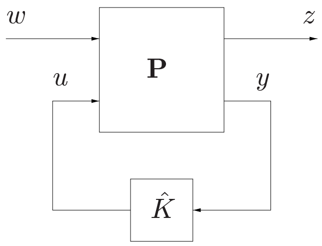
$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} \in \mathcal{RH}_\infty$$

Lemma A proves the remaining claims. Now assume the converse. Then

$$\bar{D}^{-1}\left(\bar{D} + \bar{C}(sI - \tilde{A})^{-1}\bar{B}\right)\bar{D}^{-1} \in \mathcal{RH}_\infty$$

This implies that $\bar{C}(sI - \tilde{A})^{-1}\bar{B} \in \mathcal{RH}_\infty$. Because of the stabilizability and detectability assumptions $(\tilde{A}, \bar{B}, \bar{C})$ is also stabilizable and detectable. Therefore $\tilde{A}$ must have all its eigenvalues in the open left half-pane.

Generalized Regulator Problem



$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}; \qquad u = \hat{K}y$$

Remark: The internal stablity analysis diagram is a special case with $P_{yu} = P$, $P_{yw} = \begin{bmatrix} P & I \end{bmatrix}$, and $w^T = \begin{bmatrix} w_1^T & w_2^T \end{bmatrix}$.

Definition A

A proper system P is said to be **stabilizable** (through output feedback) if there exists a proper $\hat{K}$ such that the above configuration is well-posed and internally stable. Any such controller $\hat{K}$ is said to be **admissible**.

Realizations

Let

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}$$

be a realization of $\mathbf{P}$ and let

$$\begin{bmatrix} \dot{\hat{x}} \\ u \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \end{bmatrix}$$

be a realization of $\hat{K}$.

Remark: Well-posedness and internal stability are only concerned with (A, B_2, C_2, D_{22}) and $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. Why?

Lemma B

$\mathbf{P}$ is stabilizable if and only if (A, B_2) is stabilizable and (C_2, A) is detectable.

Proof: If (A, B_2) is stabilizable and (C_2, A) is detectable, then there is an observerbased state-feedback that achieves internal stability, and hence is $\mathbf{P}$ is stabilizable. If $\mathbf{P}$ is stabilizable, then there exists a controller that achieves internal stability and (A, B_2) is stabilizable and (C_2, A) is detectable by Lemma A.

Remark: Hence it is no loss in generality to assume that (A, B_2) is stabilizable and (C_2, A) is detectable when describing the set of admissible controllers, since otherwise the set would be empty.

Youla Parametrization

Assume that (A, B_2) is stabilizable and that (C_2, A) is detectable. Then all admissible controllers can be represented on the form

$$\begin{cases} \dot{\widehat{x}} &= A\widehat{x} + B_2 u + Ke \\ u &= r - L\widehat{x} \\ e &= y - C_2\widehat{x} - D_{22}u \end{cases}$$

where $r = Q(s)e$ for $Q \in \mathcal{RH}_\infty$ with $I - D_{22}Q(\infty)$ invertible. The closed loop input-output map always takes the form

$$T_{zw}(s) + T_{zr}(s)Q(s)T_{ew}(s),$$

where $T_{zw}, T_{zr}, T_{ew} \in \mathcal{RH}_\infty$ are defined by the plant and K and L.

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Proof in easy direction

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \\ z \\ e \end{bmatrix} = \begin{bmatrix} A-B_2L & B_2L & B_1 & B_2 \\ 0 & A-KC_2 & B_1-KD_{21} & 0 \\ C_1-D_{12}L & D_{12}L & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \\ w \\ r \end{bmatrix}$$

Note that $w \equiv 0$ and $\tilde{x}(0) = 0$ gives $\tilde{x} \equiv 0$ and $e \equiv 0$, so the transfer function from r to e is zero. Therefore

$$\begin{bmatrix} z \\ e \end{bmatrix} = \begin{bmatrix} T_{zw} & T_{zr} \\ T_{ew} & 0 \end{bmatrix} \begin{bmatrix} w \\ r \end{bmatrix}.$$

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Proof in hard direction

Assume that $D_{22} = 0$. This can be achieved by the loop transformation $\bar{y} = y - D_{22}u$.

Consider arbitrary stabilizing controller $\hat{K}$ with realization

$$\begin{bmatrix} \dot{\xi} \\ u \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \xi \\ y \end{bmatrix}$$

Since $D_{22} = 0$ it holds that

$$\tilde{A} = \begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix}$$

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Proof continued

Now $u = \hat{K}(s)y$ should for some Q be generated by

$$\begin{cases} \begin{bmatrix} u \\ e \end{bmatrix} = F(s) \begin{bmatrix} y \\ r \end{bmatrix} \\ r = Q(s)e \end{cases}$$

where the realization of F is given by

$$\begin{bmatrix} \dot{\widehat{x}} \\ u \\ e \end{bmatrix} = \begin{bmatrix} A - B_2L - KC_2 & K & B_2 \\ -L & 0 & I \\ -C_2 & I & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ y \\ r \end{bmatrix}$$

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Proof continued

Alternatively $r = Qe$ should be generated by

$$\begin{cases} \begin{bmatrix} y \\ r \end{bmatrix} = F^{-1}(s) \begin{bmatrix} u \\ e \end{bmatrix} \\ u = \hat{K}(s)y \end{cases}$$

where the realization of F^{-1} is given by

$$\begin{bmatrix} \dot{\widehat{x}} \\ y \\ r \end{bmatrix} = \begin{bmatrix} A & B_2 & K \\ C_2 & 0 & I \\ L & I & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ u \\ e \end{bmatrix}$$

Proof continued

Combining the realizations for $\hat{K}$ and F^{-1} yields the following realization for Q:

$$\begin{bmatrix} \dot{\widehat{x}} \\ \dot{\xi} \\ r \end{bmatrix} = \begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} & K + B_2\hat{D} \\ \hat{B}C_2 & \hat{A} & \hat{B} \\ L + \hat{D}C_2 & \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \xi \\ e \end{bmatrix}$$

where we recognize the closed loop system matrix $\tilde{A}$. Hence Q is stable and proper.

Generalized Bezout Identity

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} \begin{bmatrix} r \\ e \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \dot{\widehat{x}} \\ u \\ y \end{bmatrix} = \begin{bmatrix} A - B_2L & B_2 & K \\ -L & I & 0 \\ C_2 & 0 & I \end{bmatrix} \begin{bmatrix} \widehat{x} \\ r \\ e \end{bmatrix}$$

$$\begin{bmatrix} r \\ e \end{bmatrix} = \begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) \\ -\widetilde{N}(s) & \widetilde{M}(s) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \dot{\widehat{x}} \\ r \\ e \end{bmatrix} = \begin{bmatrix} A - KC_2 & B_2 & K \\ L & I & 0 \\ -C_2 & 0 & I \end{bmatrix} \begin{bmatrix} \widehat{x} \\ u \\ y \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \widetilde{X}(s) & -\widetilde{Y}(s) \\ -\widetilde{N}(s) & \widetilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix}$$

Definition 5.3: Coprimeness over $\mathcal{RH}_\infty$

$M, N \in \mathcal{RH}_\infty$ are said to be **right coprime over** $\mathcal{RH}_\infty$ if there exist $X_r, Y_r \in \mathcal{RH}_\infty$ such that

$$X_rM + Y_rN = I$$

$\tilde{M}, \tilde{N} \in \mathcal{RH}_\infty$ are said to be **left coprime over** $\mathcal{RH}_\infty$ if there exist $X_l, Y_l \in \mathcal{RH}_\infty$ such that

$$\tilde{M}X_l + \tilde{N}Y_l = I$$

Coprime Factorizations

A **right-coprime factorization** (rcf) of a proper real transfer function P is a factorization

$$P = NM^{-1}$$

where M and N are right coprime over $\mathcal{RH}_\infty$.

A **left-coprime factorization** (lcf) of a proper real transfer function P is a factorization

$$P = \tilde{M}^{-1}\tilde{N}$$

where $\tilde{M}$ and $\tilde{N}$ are left coprime over $\mathcal{RH}_\infty$.

Double Coprime Factorizations

A proper real rational transfer function is said to have a **double coprime factorization** if there exist a rcf $P = NM^{-1}$ and a lcf $P = \tilde{M}^{-1}\tilde{N}$ and $X_r, Y_r, X_l, Y_l \in \mathcal{RH}_\infty$ such that

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = I$$

Theorem 5.9

From the Youla parameterization

$$e = -\tilde{N}u\tilde{M}y$$
$$u = Mr + Ye$$
$$y = Nr + Xe$$

For $e = 0$ it then holds that

$$y = \tilde{M}^{-1}\tilde{N}u = NM^{-1}u$$

So we have a double coprime factorization of P_{yu}.

Corollary 5.11

From the Youla parameterization

$$r = \tilde{X}u - \tilde{Y}y$$
$$u = Mr + Ye$$
$$y = Nr + Xe$$

For $r = 0$ it then holds that

$$u = \tilde{X}^{-1}\tilde{Y}y = YX^{-1}y$$

So we have a double coprime factorization of $\hat{K}$.

Lemma 5.10: On Internal Stability

Consider any rcf's and lcf's of P and $\hat{K}$:

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$
$$K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$$

Then the following conditions are equivalent

Lemma 5.10 continued

1. $\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} \in \mathcal{RH}_\infty$

2. $\begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1} \in \mathcal{RH}_\infty$

3. $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}^{-1} \in \mathcal{RH}_\infty$

4. $(\tilde{M}V - \tilde{N}U)^{-1} \in \mathcal{RH}_\infty$

5. $(\tilde{V}M - \tilde{U}N)^{-1} \in \mathcal{RH}_\infty$

Proof: See Zhou, Doyle and Glover (1996) and fill in the gaps.

Alternative Definition of Coprimeness over $\mathcal{RH}_\infty$

For $M, N \in \mathcal{RH}_\infty$ it holds that they are right coprime if and only if for any common right divisor R it holds that $R^{-1} \in \mathcal{RH}_\infty$

Proof: See e.g. Vidyasagar (1985): Control System Synthesis: A Factorization Approach, The MIT Press, Cambridge, Massachusetts.

Remark: We see that unimodularity is replaced with invertability in $\mathcal{RH}_\infty$ when we consider transfer functions in $\mathcal{RH}_\infty$ instead of polynomial matrices.