Lecture 6

- Norms
- Frequency curves
- Limits of performance
- Bode gain-phase relation

$\mathcal{L}_2$-Spaces

$\mathcal{L}_2^{m\times n}(I)$ for $I \subset \mathbf{R}$ is a **Hilbert space** that consists of all square integrable and Lebesgue measurable $m \times n$ matrix-valued functions defined on an interval $I \subset \mathbf{R}$ with the **inner product**

$$< f, g >= \int_I \mathrm{tr}[f^*(t)g(t)]dt$$

for $f, g \in \mathcal{L}_2^{m\times n}(I)$. The corresponding **norm** is defined by

$$\|f\|_2^2 =< f, f >$$

Remark 1: $\mathcal{L}_2^n(I)$ has inner product $< f, g >= \int_I f(t)^* g(t)dt$.

Remark 2: $\mathcal{L}_2^n$ denotes $\mathbf{L}_2^n(-\infty, \infty)$ and $\mathcal{L}_{2+}^n$ denotes $\mathbf{L}_2^n[0, \infty)$.

$\mathcal{L}_2$-Spaces on the $i\omega$-axis

$\mathcal{L}_2^{m\times n}(i\mathbf{R})$ is a **Hilbert space** that consists of all square integrable and Lebesgue measurable $m \times n$ matrix-valued functions defined on $i\mathbf{R}$ with the **inner product**

$$< \mathbf{f}, \mathbf{g} >= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}[\mathbf{f}^*(i\omega)\mathbf{g}(i\omega)]d\omega$$

for $\mathbf{f}, \mathbf{g} \in \mathcal{L}_2^{m\times n}(i\mathbf{R})$. The corresponding **norm** is defined by

$$\|\mathbf{f}\|_2^2 =< \mathbf{f}, \mathbf{f} >$$

Remark 1: Boldface letters are used here to emphasize that the functions are defined for complex-valued arguments.

Remark 2: Often we will use uppercase letters for matrices and lowercase letters for vectors.

Analytic Functions and the Laplace Transform

Definition: Let $S \subset \mathbf{C}$ be an open set, and let $\mathbf{f}(s)$ be a complex valued function defined on S. Then $\mathbf{f}(s)$ is said to be **analytic at a point** z_0 if it is differentiable at z_0 and also at each point in some neighborhood of z_0. The function $\mathbf{f}(s)$ is **analytic** in S if it is analytic at each point $z_0 \in S$.

Laplace transform: For a function x on $[0, \infty)$ introduce

$$\mathbf{x}(s) = (\mathcal{L}x)(s) \quad := \quad \int_0^\infty x(t)e^{-st}dt, \quad \mathrm{Re}(s) > \sigma$$

If $x \in \mathcal{L}_{2+}^n$, then $\sigma \leq 0$.

Remark: $\mathbf{x}(s)$ is analytic in $\mathrm{Re}(s) > \sigma$.

$\mathcal{H}_2$

$\mathcal{H}_2$ is a subspace of $\mathcal{L}_2^{m\times n}(i\mathbf{R})$ with matrix-valued functions $\mathbf{f}(s)$ that are **analytic** in $\mathrm{Re}(s) > 0$. The corresponding **norm** is defined as

$$\|\mathbf{f}\|_2^2 = \sup_{\sigma>0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}[\mathbf{f}^*(\sigma+i\omega)\mathbf{f}(\sigma+i\omega)]d\omega\right\}$$

It can be shown that (see Francis 1987)

$$\|\mathbf{f}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}[\mathbf{f}^*(i\omega)\mathbf{f}(i\omega)]d\omega$$

$\mathcal{RH}_2$ is a subspace of $\mathcal{H}_2$ with **strictly proper real rational stable** matrix-valued functions.

Parseval's Relation

For $f \in \mathcal{L}_{2+}$ and $\mathbf{f} = \mathcal{L}f \in \mathcal{H}_2$ it holds that

$$\|f\|_2 = \|\mathbf{f}\|_2$$

$\mathcal{L}_\infty$

$\mathcal{L}_\infty(j\mathbf{R})$ or simply $\mathcal{L}_\infty$ is a **Banach** space of matrix valued functions that are **bounded** on $i\mathbf{R}$ with norm

$$\|\mathbf{f}\|_\infty = \sup_{\omega\in\mathbf{R}}\bar{\sigma}[\mathbf{f}(i\omega)]$$

$\mathcal{H}_\infty$ is a subspace of $\mathcal{L}_\infty$ with functions that are **analytic and bounded** in $\mathrm{Re}(s) > 0$ with norm

$$\|\mathbf{f}\|_\infty = \sup_{\mathrm{Re}(s)>0}\bar{\sigma}[\mathbf{f}(s)] = \sup_{\omega\in\mathbf{R}}\bar{\sigma}[\mathbf{f}(j\omega)]$$

See Boyd and Desoer (1985) for a proof of the second equality.

$\mathcal{RH}_\infty$ is a subspace of $\mathcal{H}_\infty$ with **proper real rational stable** matrix-valued functions.

Operator

A **bounded linear operator** is a linear map that has a bounded **induced norm**

$$\|M\| \quad := \quad \sup_{x\neq 0}\frac{\|Mx\|}{\|x\|} = \sup_{\|x\|=1}\|Mx\| = \sup_{\|x\|\leq 1}\|Mx\|$$

Theorem 4.4 Let $\mathbf{G} \in \mathcal{H}_\infty$ be an $m \times n$ transfer function and consider it as a bounded linear operator from $\mathcal{H}_2^m$ to $\mathcal{H}_2^n$. Then $\|\mathbf{G}\| = \|\mathbf{G}\|_\infty$.

Remark: If $\mathbf{G} = \mathcal{L}g$, then by Parseval's formula $\|\mathbf{G}\|_\infty$ is also the induced norm for the convolution operator defined by

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

where $u \in \mathcal{L}_{2+}^m$ and $y \in \mathcal{L}_{2+}^n$.

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Proof

By definition

$$\|\mathbf{G}\| = \sup\left\{\|\mathbf{Gu}\|_2 : \ \mathbf{u} \in \mathcal{L}_2^m, \ \|\mathbf{u}\|_2 \leq 1\right\}$$

First notice that $\|\mathbf{G}\|_\infty$ is an upper bound for the induced operator norm, since

$$\begin{aligned}\|\mathbf{Gu}\|_2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{u}^*(j\omega)\mathbf{G}^*(j\omega)\mathbf{G}(j\omega)\mathbf{u}(j\omega)d\omega \\ &\leq \|\mathbf{G}\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{u}^*(j\omega)\mathbf{u}(j\omega)d\omega = \|\mathbf{G}\|_\infty^2\|\mathbf{u}\|_2^2\end{aligned}$$

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Proof continued

Let ω_0 be such that $\|\mathbf{G}(i\omega)\|_\infty = \bar{\sigma}[\mathbf{G}(i\omega_0)]$ and use the SVD of $\mathbf{G}(i\omega_0)$ to write

$$\mathbf{G}(i\omega_0) = \bar{\sigma}(i\omega_0)\mathbf{u}_1(i\omega_0)\mathbf{v}_1^*(i\omega_0) + \sum_{j=2}^{r}\sigma_j(i\omega_0)\mathbf{u}_j(i\omega_0)\mathbf{v}_j^*(i\omega_0)$$

where $\mathbf{u}_1(i\omega_0)$ and $\mathbf{v}_1(i\omega_0)$ have unit length. Write for $s = i\omega_0$

$$\mathbf{v}_1(i\omega_0) = \begin{bmatrix} \alpha_1\frac{\beta_1 - s}{\beta_1 + s} \\ \vdots \\ \alpha_m\frac{\beta_m - s}{\beta_m + s}\end{bmatrix}$$

with 1 replacing $\frac{\beta_m - s}{\beta_m + s}$ in case β_i has to be infinite, and where $0 < \beta_i < \infty$.

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Proof continued

Let $\hat{\mathbf{f}}$ be a scalar function such that

$$|\hat{\mathbf{f}}(i\omega)| = \begin{cases}\sqrt{\pi/2\epsilon} & \text{if } |\omega - \omega_0| < \epsilon \text{ or } |\omega + \omega_0| < \epsilon \\ 0 & \text{otherwise}\end{cases}$$

and define

$$\mathbf{u}(s) = \begin{bmatrix} \alpha_1\frac{\beta_1 - s}{\beta_1 + s} \\ \vdots \\ \alpha_m\frac{\beta_m - s}{\beta_m + s}\end{bmatrix}\hat{\mathbf{f}}(s)$$

Remark: $\mathbf{u}(s)$ is almost in $\mathcal{H}_2$, at least when $\epsilon \to 0$. A formal proof of this theorem is outside the scope of the course.

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Proof continued

Then using the fact that $\mathbf{G}(i\omega_0)\mathbf{v}_1(i\omega_0) = \bar{\sigma}^2(i\omega_0)\mathbf{u}_1(i\omega_0)$:

$$\begin{aligned}\|\mathbf{Gu}\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{u}^*(i\omega)\mathbf{G}^*(i\omega)\mathbf{G}(i\omega)\mathbf{u}(i\omega)d\omega \\ &\approx \frac{1}{2\pi}\left[\bar{\sigma}^2(-i\omega_0)\pi + \bar{\sigma}^2(i\omega_0)\pi\right] = \bar{\sigma}^2(i\omega_0) = \|\mathbf{G}\|_\infty^2\end{aligned}$$

Computing the Norm of $\mathbf{G} \in \mathcal{H}_2$

Under mild conditions on

$$\|\mathbf{G}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}[\mathbf{G}^*(i\omega)\mathbf{G}(i\omega)]d\omega = \frac{1}{2\pi i}\oint \mathrm{tr}[\mathbf{G}^T(-s)\mathbf{G}(s)]ds$$

where the last integral is a contour integral along the imaginary axis, and around an infinite semi-circle in the left half-plane.

The mild conditions have to be such that the integral from this semi-circle equals zero, basically "strict properness".

By the residue theorem $\|\mathbf{G}\|_2^2$ equals the sum of the residues of the singularities of $\mathbf{G}^T(-s)\mathbf{G}(s)$ in the left half-plane.

Remark: For $\mathbf{G} \in \mathcal{RH}_2$ the mild conditions are satisfied.

Computing the Norm of $\mathbf{G} \in \mathcal{RH}_2$

Lemma 4.6 Let (A, B, C) be a realization of $\mathbf{G} \in \mathcal{RH}_2$ such that A has all its eigenvalues in the open left half-plane. Then

$$\|\mathbf{G}\|_2^2 = \mathrm{tr}(B^*W_oB) = \mathrm{tr}(CW_cC^*)$$

where W_c and W_o are the controllability and and observability Gramians.

Proof: The impulse response for $\mathbf{G}$ is given by $g(t) = Ce^{At}B$ and hence by Parseval's formula

$$\begin{aligned}\|\mathbf{G}\|_2^2 &= \int_0^{\infty} \mathrm{tr}[g^*(t)g(t)]dt = \int_0^{\infty} \mathrm{tr}[g(t)g^*(t)]dt \\ &= \mathrm{tr}\int_0^{\infty} B^*e^{A^*t}C^*Ce^{At}Bdt = \mathrm{tr}\int_0^{\infty} Ce^{At}BB^*e^{A^*t}C^*dt\end{aligned}$$

Remark: Remember that Gramians satisfy Lyapunov equations.

Computing the Norm of $\mathbf{G} \in \mathcal{RL}_\infty$

Lemma 4.6 Let $\gamma > 0$ and let (A, B, C, D) be a realization of $\mathbf{G} \in \mathcal{RL}_\infty$. Then $\|\mathbf{G}\|_\infty < \gamma$ if and only if $\bar{\sigma}(D) < \gamma$ and

$$H = \begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^* \end{bmatrix}$$

has no eigenvalues on the imaginary axis, where $R = \gamma^2 I - D^*D$.

Proof: See Zhou, Doyle, and Glover (1996).

Remark: Bisectioning can be used to find the smallest γ.

Transmission Zeros Revisited

Assumption 1 (A1): (A, B, C, D) is a minimal realization of $\mathbf{G}(s)$ where

$$\mathbf{P}(s) = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$$

has full normal row and column rank.

Lemma: Under (A1) $\mathbf{G}(s)$ is square and invertible for almost all s.

Proof: Exercise.

Remark: Under (A1) s_0 is a transmission zero of $\mathbf{G}(s)$ if and only if there exists $\xi^T = \begin{bmatrix}\zeta^T & \eta^T\end{bmatrix} \neq 0$ such that

$$\mathbf{P}(s_0)\xi = 0$$

More on Transmission Zeros

Assumption 2 (A2): Assume A1 holds and that if s_0 is a transmission zero of $\mathbf{G}(s)$ it is not an eigenvalue of A.

Lemma: Assume (A2) holds. Then s_0 is a transmission zero of $\mathbf{G}(s)$ if and only if there is $\eta \neq 0$ such that $\mathbf{G}(s_0)\eta = 0$.

Proof: Exercise

Definition: A transmission zero s_0 is called a **non-minimum phase** zero if it is in the open right half plane and otherwise it is called a **minimum phase** zero.

Remarks

Assume that A2 holds. Then:

- s_0 is a transmission zero of $\mathbf{G}(s)$ if and only if there is $w \neq 0$ such that $w^*\mathbf{G}(s_0) = 0$.
- The vectors η and w are called the input zero direction and the output zero direction (of the transfer function), respectively.
- s_0 is a pole of $\mathbf{G}(s)$ if and only if it is a transmission zero of $\mathbf{G}^{-1}(s)$.
- The input and output zero directions of $\mathbf{G}^{-1}(s)$ are called the input and output pole directions of $\mathbf{G}(s)$, respectively.
- The direction vectors can be normalized to unit length.

Input Factorization

Assume that (A2) holds and let s_i, $i = 1, \ldots, k$ be the non-minimum phase zeros of $\mathbf{G}(s)$ and let η_i, $\eta_i^*\eta_i = 1$ be the input directions generated from the following iterative procedure

Let $B^{(0)} := B$. Repeat for $i = 1, \ldots, k$:

$$\begin{bmatrix} A - s_iI & B^{(i-1)} \\ C & D \end{bmatrix} \begin{bmatrix} \zeta_i \\ \eta_i \end{bmatrix} = 0; \qquad B^{(i)} := B^{(i-1)} - 2(\mathrm{Re}s_i)\zeta_i\eta_i^*$$

Define

$$\mathbf{B}_i(s) = I - \frac{2\mathrm{Re}s_i}{s + \bar{s}_i}\eta_i\eta_i^*; \quad \mathbf{G}_m(s) = C(sI - A)^{-1}B^{(k)} + D$$

Then $\mathbf{G}(s) = \mathbf{G}_m(s)\mathbf{B}_k(s)\cdots\mathbf{B}_1(s)$.

Interpolation Constraint

For a **loop transfer function** $\mathbf{L}(s)$ the **sensitivity function** is defined as

$$\mathbf{S}(s) = (I + \mathbf{L}(s))^{-1}$$

Lemma: Assume that (A2) holds for $\mathbf{L}(s)$. Then p_0 is a pole of $\mathbf{L}(s)$ with input pole direction η if and only if p_0 is a transmission zero of $\mathbf{S}(s)$ with input zero direction η.

Proof: Follows from the identity

$$(I + \mathbf{L}(s))^{-1} = (I + \mathbf{L}^{-1}(s))^{-1}\mathbf{L}^{-1}(s)$$

Remark: In the remainder of this lecture it is assumed that (A2) holds for $\mathbf{L}(s)$.

Corollary

If $\mathbf{L}(s)$ has poles p_i in $\mathbf{C}_+$ with input pole directions η_i, then we can factorize $\mathbf{S}(s)$ as $\mathbf{S}(s) = \mathbf{S}_m(s)\mathbf{B}_k(s)\cdots\mathbf{B}_1(s)$, where $\mathbf{S}_m(s)$ has no zeros in $\mathbf{C}_+$, and where

$$\mathbf{B}_i(s) = I - \frac{2\mathrm{Re}p_i}{s+\bar{p}_i}\eta_i\eta_i^*$$

Assumption 6.1 Roll-off of more than one pole-zero excess, i.e.

$$\lim_{R\to\infty}\sup_{s\in\bar{\mathbf{C}}_+,\ |s|\geq R} R\bar{\sigma}(L(s)) = 0$$

Theorem 6.1: Bode's Integral

Let $\mathbf{S}(s)$ be factorized as in the previous corollary and suppose that Assumption 6.1 holds. Then

$$\int_0^\infty \ln\bar{\sigma}(\mathbf{S}(i\omega))d\omega \geq \pi\lambda_{\max}\left(\sum_{j=1}^k(\mathrm{Re}p_j)\eta_j\eta_j^*\right)$$

Proof: See Chen (1995)

Remark: Notice that

$$\int_0^\infty \ln|\mathbf{S}(i\omega)|d\omega = \pi\sum_{j=1}^k \mathrm{Re}p_j$$

for the **scalar** case.

Special Cases

If $\eta_i^*\eta_j = 0$ for $i \neq j$, then

$$\int_0^\infty \ln\bar{\sigma}(\mathbf{S}(i\omega))d\omega \geq \pi\max_i\mathrm{Re}p_j$$

and if $\eta_i = \eta$ for all i, then

$$\int_0^\infty \ln\bar{\sigma}(\mathbf{S}(i\omega))d\omega \geq \pi\sum_{j=1}^k \mathrm{Re}p_j$$

Theorem 6.3: Poisson Integral Relation

Let $\mathbf{S}(s)$ be factorized as before and assume that

$$\lim_{R\to\infty}\max_{\phi\in[-\pi/2,\pi/2]}\frac{|\ln\bar{\sigma}(\mathbf{S}(Re^{j\phi}))|}{R} = 0$$

Then for any transmission zero $s_0 = x_0 + iy_0 \in \mathbf{C}_+$ of $\mathbf{L}(s)$ with output zero direction w, $w^*w = 1$,

$$\int_{-\infty}^\infty \ln\bar{\sigma}(\mathbf{S}(i\omega))\frac{x_0}{x_0^2+(\omega-y_0)^2}d\omega \geq \pi\ln\bar{\sigma}(\mathbf{S}_m(s_0)) \geq \pi\ln\bar{\sigma}(\mathbf{S}(s_0))$$

$$\int_{-\infty}^\infty \ln\bar{\sigma}(\mathbf{S}(i\omega))\frac{x_0}{x_0^2+(\omega-y_0)^2}d\omega \geq \pi\ln\|w^*\mathbf{B}_1^{-1}(s_0)\cdots\mathbf{B}_k^{-1}(s_0)\|$$

Proof: See Chen (1995)

Single-Input Single-Output Case

For single-input single-output systems

$$\|w^*\mathbf{B}_1^{-1}(s_0)\cdots\mathbf{B}_k^{-1}(s_0)\| = \prod_{i=1}^{k}\left|\frac{s_0+\bar{p}_i}{s_0-p_i}\right|$$

and

$$\int_{-\infty}^{\infty}\ln|\mathbf{S}(i\omega)|\frac{x_0}{x_0^2+(\omega-y_0)^2}d\omega = \pi\ln\prod_{i=1}^{k}\left|\frac{s_0+\bar{p}_i}{s_0-p_i}\right|$$

Limitations imposed by Non-Minimum-Phase Zeros

Corollary 6.4: Assume that the assumptions in Theorem 6.3 hold. Then with the notations in that theorem define

$$\theta(s_0) = \int_{-\omega_L}^{\omega_L}\frac{x_0}{x_0^2+(\omega-y_0)^2}d\omega$$

Suppose that $\bar{\sigma}(\mathbf{S}(i\omega)) \le M_L < 1, \quad \forall\omega\in[-\omega_L,\omega_L]$. Then

$$\|\mathbf{S}\|_\infty \ge \left(\frac{1}{M_L}\right)^{\frac{\theta(s_0)}{\pi-\theta(s_0)}}\|w^*\mathbf{B}_1^{-1}(s_0)\cdots\mathbf{B}_k^{-1}(s_0)\|^{\frac{\theta(s_0)}{\pi-\theta(s_0)}}$$

$$\|\mathbf{S}\|_\infty \ge \left(\frac{1}{M_L}\right)^{\frac{\theta(s_0)}{\pi-\theta(s_0)}}(\bar{\sigma}(\mathbf{S}(s_0)))^{\frac{\theta(s_0)}{\pi-\theta(s_0)}}$$

Proof: Follows from Th. 6.3, see Zhou, Doyle, and Glover (1996).

Limitations imposed by Unstable Poles

Corollary 6.5: Suppose that

$$\bar{\sigma}(\mathbf{S}(i\omega)) \le M_L < 1, \quad \forall\omega\in[-\omega_L,\omega_L]$$

$$\bar{\sigma}(\mathbf{L}(i\omega)) \le \frac{M_H}{\omega^{1+k}} \le \epsilon < 1, \quad \forall\omega\in[-\omega_H,\infty)$$

where $\omega_L < \omega_H$. Then with the notations of Th. 6.1

$$\max_{\omega\in[\omega_L,\omega_H]}\bar{\sigma}(\mathbf{S}(i\omega)) \ge e^{\alpha}\left(\frac{1}{M_L}\right)^{\frac{\omega_L}{\omega_H-\omega_L}}(1-\epsilon)^{\frac{\omega_L}{k(\omega_H-\omega_L)}}$$

$$\alpha = \frac{\pi\lambda\max\left(\sum_{j=1}^{k}(\mathrm{Re}p_j)\eta_j\eta_j^*\right)}{\omega_H-\omega_L}$$

Proof: Follows from Theorem 6.1, see Zhou, Doyle, and Glover (1996).

Bode's Gain-Phase Relation

Assume that $\mathbf{L}(s)$ is a single-input single-output rational proper transfer function with no poles or zeros in the closed right half-plane and that $\log\mathbf{L}(s)/s \to 0$ as $s\to\infty$. Write

$$\log\mathbf{L}(i\omega) = A(\omega) + i\Phi(\omega)$$

$$a(u) = A(\omega_0 e^u), \quad \phi(u) = \Phi(\omega_0 e^u)$$

where $u = \log(\omega/\omega_0)$. Then

$$\Phi(\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{da(u)}{du}\log\coth\left|\frac{u}{2}\right|du$$

Proof: See Åström (1999)

Remark: An approximate version is $\Phi(\omega_0) \approx \frac{2}{\pi}\frac{da(u)}{du}$.

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Next Lecture

- Well-posedness
- Internal stability
- The Q-parameterization
- Coprime factorizations over $\mathcal{RH}_\infty$