Lecture 5

- State Space — Geometric Theory
- Invariant Zeros — Geometric Theory
- Eigenvalue assignment
- Disturbance decoupling
- Noninteracting control

$\mathcal{R}^* \subset \mathcal{V}^* \subset \mathrm{Ker} C$

It holds that $\mathcal{R}^* \subset \mathcal{V}^* \subset \mathrm{Ker} C$ and that

$\mathcal{V}^* =$ the set of x_0 that can be kept in $\mathrm{Ker} C$ for all $t \geq 0$.

$\mathcal{R}^* =$ the set of x that can be reached from $x_0 = 0$ with $x \in \mathrm{Ker} C$ for all t.

Proof: By construction both $\mathcal{R}^* \subset \mathrm{Ker} C$ and $\mathcal{V}^* \subset \mathrm{Ker} C$. Now, both $\mathcal{R}^*$ and $\mathcal{V}^*$ are (A, B)-invariant subspaces contained in $\mathrm{Ker} C$, but $\mathcal{V}^*$ is the maximal of all such subspaces. Therefore $\mathcal{R}^* \subset \mathcal{V}^*$. The remaining claims will be proven in what follows.

Theorem 1 (Rugh 19.8, Kailath 7.6.2+)

Given a realization (A, B), let F be such that $\hat{A}\mathcal{V} \subset \mathcal{V}$, where $\hat{A} = A + BF$. Then, for some state and input coordinates (z, u), the system takes the form

$$\begin{bmatrix} \dot{z} \\ y \end{bmatrix} = \left[\begin{array}{ccc|cc} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{B}_{11} & 0 \\ 0 & \hat{A}_{22} & \hat{A}_{23} & 0 & 0 \\ 0 & 0 & \hat{A}_{33} & 0 & \hat{B}_{32} \\ \hline 0 & 0 & C_3 & 0 & 0 \end{array}\right] \begin{bmatrix} z \\ u \end{bmatrix}$$

where $\mathcal{V} = \{(z_1, z_2, 0)\}$, $(\hat{A}_{11}, \hat{B}_{11})$ is controllable and $\hat{B}_{32}$ has full column rank.

Remark

Additional feedback of the form

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & 0 & F_{23} \end{bmatrix}$$

preserves the structure, while

$$\begin{aligned} \hat{A}_{11} &\to \hat{A}_{11} + \hat{B}_{11} F_{11} \\ \hat{A}_{22} &\to \hat{A}_{22} \\ \hat{A}_{33} &\to \hat{A}_{33} + \hat{B}_{32} F_{23} \end{aligned}$$

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Proof

Choose basis $e_1, \ldots, e_n$ with the first elements in $\mathcal{V}$, Then $\hat{A}\mathcal{V} \subset \mathcal{V} \subset \mathrm{Ker} C$ implies the structure

$$\begin{bmatrix} \dot{z} \\ y \end{bmatrix} = \left[\begin{array}{cc|c} \hat{A}_{11} & \hat{A}_{13} & \hat{B}_1 \\ 0 & \hat{A}_{33} & \hat{B}_3 \\ \hline 0 & C_3 & 0 \end{array}\right] \begin{bmatrix} z \\ u \end{bmatrix}$$

With new input coordinates

$$\begin{bmatrix} \dot{z} \\ y \end{bmatrix} = \left[\begin{array}{cc|cc} \hat{A}_{11} & \hat{A}_{13} & \hat{B}_{11} & \hat{B}_{12} \\ 0 & \hat{A}_{33} & 0 & \hat{B}_{32} \\ \hline 0 & C_3 & 0 & 0 \end{array}\right] \begin{bmatrix} z \\ u \end{bmatrix}$$

where $\hat{B}_{32}$ has full column rank (possibly zero rank if zero dimension)

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Proof continued

If $\hat{B}_{32}$ does not have zero dimension, then the space transformation $x \mapsto Tx$ with

$$T = \begin{bmatrix} I & -\hat{B}_{12}\hat{B}_{32}^{\dagger} \\ 0 & I \end{bmatrix}$$

removes $\hat{B}_{12}$ but leaves the structure of $\hat{A}$ and C_3 unchanged.

A final coordinates change

$$\hat{A}_{11} \rightarrow \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, \quad \hat{B}_{11} \rightarrow \begin{bmatrix} \hat{B}_{11} \\ 0 \end{bmatrix}$$

where $(\hat{A}_{11}, \hat{B}_{11})$ controllable, shows the claimed structure.

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Corollary

If $\mathcal{V} = \mathcal{V}^*$, then $\mathcal{R}^* = \{(z_1, 0, 0)\}$

Proof: Clearly $\{(z_1, 0, 0)\} \subset \mathrm{Ker} C$, and with $F = 0$ and $G = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

$$\langle \hat{A} + \hat{B}F | \mathrm{Im} \hat{B}G \rangle = \{(z_1, 0, 0)\}$$

so $\{(z_1, 0, 0)\}$ is a controllability subspace contained in $\mathrm{Ker} C$.

Let $\mathcal{R}$ be an arbitrary controllability subspace such that $\{(z_1, 0, 0)\} \subset \mathcal{R} \subset \mathrm{Ker} C$. Notice that $\mathcal{R} \subset \mathcal{V}^*$.

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Proof continued

With

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \Rightarrow \hat{B}G = \begin{bmatrix} \hat{B}_{11} & 0 \\ 0 & 0 \\ 0 & \hat{B}_{32} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} \hat{B}_{11}G_1 \\ 0 \\ \hat{B}_{32}G_2 \end{bmatrix}$$

So for some F and G

$$\{(z_1, z_2, 0)\} = \mathcal{V}^* \supset \mathcal{R} = \langle \hat{A} + \hat{B}F | \mathrm{Im} \hat{B}G \rangle \supset \mathrm{Im} \begin{bmatrix} \hat{B}_{11}G_1 \\ 0 \\ \hat{B}_{32}G_2 \end{bmatrix}$$

implying $\hat{B}_{23}G_2 = 0$. Since $\hat{B}_{23}$ is full column rank $G_2 = 0$.

Proof continued

$$\hat{A} + \hat{B}F = \begin{bmatrix} \hat{A}_{11} + \hat{B}_{11}F_{11} & \hat{A}_{12} + \hat{B}_{11}F_{12} & \hat{A}_{13} + \hat{B}_{11}F_{13} \\ 0 & \hat{A}_{22} & \hat{A}_{23} \\ \hat{B}_{23}F_{21} & \hat{B}_{23}F_{22} & \hat{A}_{33}\hat{B}_{23}F_{23} \end{bmatrix}$$

Since $\mathcal{R} \supset \{(z_1, 0, 0)\}$ and since $\mathcal{R}$ is $(\hat{A}, \hat{B})$-invariant it must hold that $\hat{B}_{23}F_{21} = 0 \Rightarrow F_{21} = 0$. Therefore

$$\mathcal{R} = \langle \hat{A} + \hat{B}F | \mathrm{Im} \hat{B}G \rangle = \mathrm{Im} \begin{bmatrix} \hat{B}_{11}G_1 & (\hat{A}_{11} + \hat{B}_{11}F_1)\hat{B}_{11}G_1 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \end{bmatrix}$$

So $\mathcal{R} \subset \{(z_1, 0, 0)\}$, but also $\mathcal{R} \supset \{(z_1, 0, 0)\}$ by construction, and therefore $\mathcal{R}^* = \{(z_1, 0, 0)\}$.

Theorem 19.6

Let F be such that $(A + BF)\mathcal{V}^* \subset \mathcal{V}^*$. Then

$$\mathcal{R}^* = \langle A + BF | \mathcal{B} \cap \mathcal{V}^* \rangle$$

Proof: For any F as above the algorithm of Theorem 1 with $\mathcal{V} = \mathcal{V}^*$ can be used to transform (A, B) to $(\hat{A}, \hat{B})$. Then

$$\hat{\mathcal{B}} \cap \mathcal{V}^* = \mathrm{Im} \begin{bmatrix} \hat{B}_{11} & 0 \\ 0 & 0 \\ 0 & \hat{B}_{32} \end{bmatrix} \cap \mathrm{Im} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} = \mathrm{Im} \begin{bmatrix} \hat{B}_{11} \\ 0 \\ 0 \end{bmatrix}$$

Proof continued

Since $(\hat{A}_{11}, \hat{B}_{11})$ is controllable

$$\langle \hat{A} | \hat{\mathcal{B}} \cap \mathcal{V}^* \rangle = \mathrm{Im} \begin{bmatrix} \hat{B}_{11} & \hat{A}_{11}\hat{B}_{11} & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \end{bmatrix}$$

$$= \{(z_1, 0, 0)\} = \mathcal{R}^*$$

The proof is completed by applying the inverse of the state transformation of Theorem 1.

Theorem 2: On Invariant Zeros

Consider a state-space realization (A, B, C). Then the following holds:

1. $\mathcal{R}^* = \{0\}$ if and only if

$$\max_{s \in \mathbf{C}} \mathrm{rank} \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix} = m + n$$

2. s_0 is an invariant zero of (A, B, C), i.e.

$$\mathrm{rank} \begin{bmatrix} s_0 I - A & B \\ -C & 0 \end{bmatrix} < \max_{s \in \mathbf{C}} \mathrm{rank} \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}$$

if and only if it is an eigenvalue of $\hat{A}_{22}$ as defined in Theorem 1 $(\mathcal{V} = \mathcal{V}^*)$.

Lemma 1

State feedback and state transformations do not change the invariant zeros of a state-space realization (A, B, C).

Proof: Combine the identities

$$\begin{bmatrix} sI-(A+BF) & B \\ -C & 0 \end{bmatrix} = \begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -F & I \end{bmatrix}$$

$$\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} sI-\hat{A} & \hat{B} \\ -\hat{C} & 0 \end{bmatrix} = \begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix}\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}$$

with the definition of invariant zeros in Lecture 3.

Lemma 2

Assume that B has full column rank (0 dimension and 0 rank possible). If $\mathcal{V}^* = \{0\}$, then

$$\mathrm{rank}\begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} = n+m \text{ for all } s.$$

Proof: Assume that there exists an s_0 such that the rank is less than $n+m$. Then there exist $\begin{bmatrix} x_0^T & u_0^T \end{bmatrix}^T \neq 0$ such that

$$\begin{bmatrix} s_0I-A & B \\ -C & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

If $u_0 = 0$, the (C, A) is not observable and hence $\mathcal{V}^* \neq \{0\}$, which is a contradiction.

Proof continued

Now consider the case when $u_0 \neq 0$. Assume that $x_0 = 0$. Then $Bu_0 = 0$ implies that B is not full column rank, which is a contradiction.

So $x_0 \neq 0$. Since x_0 has full column rank there is an F such that $u_0 = Fx_0$. This implies that

$$\begin{aligned}(A-BF)x_0 &= s_0x_0 \\ Cx_0 &= 0\end{aligned}$$

so $\mathrm{Im}x_0$ is an (A, B)-invariant subspace in $\mathrm{Ker}C$, and hence $\mathcal{V}^* \neq \{0\}$, which is a contradiction.

Proof of Theorem 2

The rank conditions are unaffected by the elementary row and column operations from the proof of Theorem 1 (using $\mathcal{V} = \mathcal{V}^*$):

$$\begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} \sim \begin{bmatrix} sI-\hat{A}_{11} & -\hat{A}_{12} & -\hat{A}_{13} & \hat{B}_{11} & 0 \\ 0 & sI-\hat{A}_{22} & -\hat{A}_{23} & 0 & 0 \\ 0 & 0 & sI-\hat{A}_{33} & 0 & \hat{B}_{32} \\ 0 & 0 & -C_3 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} sI-\hat{A}_{11} & \hat{B}_{11} & -\hat{A}_{12} & -\hat{A}_{13} & 0 \\ 0 & 0 & sI-\hat{A}_{22} & -\hat{A}_{23} & 0 \\ 0 & 0 & 0 & sI-\hat{A}_{33} & \hat{B}_{32} \\ 0 & 0 & 0 & -C_3 & 0 \end{bmatrix}$$

Proof continued

$\begin{bmatrix} sI-\hat{A}_{11} & \hat{B}_{11} \end{bmatrix} \sim \begin{bmatrix} I & 0 \end{bmatrix}$, since $(\hat{A}_{11}, \hat{B}_{11})$ is controllable.

$\begin{bmatrix} sI-\hat{A}_{33} & \hat{B}_{32} \\ -C_3 & 0 \end{bmatrix} \sim \begin{bmatrix} I \\ 0 \end{bmatrix}$, since otherwise $\mathcal{V}^*$ would not be maximal by Lemma 2.

Therefore

$$\begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} \sim \begin{bmatrix} I \quad 0 & * & * \\ 0 & sI-\hat{A}_{22} & * \\ 0 & 0 & \begin{matrix} I \\ 0 \end{matrix} \end{bmatrix}$$

Disturbance Decoupling

The linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) \\ y(t) &= Cx(t) \end{aligned}$$

with disturbance w, is said to be **disturbance decoupled** by

$$u(t) = Fx(t) + v(t)$$

if regardless of the reference signal v and the initial state $x(0)$, the output y of the closed loop state equation

$$\begin{aligned} \dot{x}(t) &= (A+BF)x(t) + Bv(t) + Ew(t) \\ y(t) &= Cx(t) \end{aligned}$$

is not influenced by w.

Theorem 19.2 Disturbance Decoupling

There exists a state feedback gain F that achieves disturbance decoupling if and only if

$$\mathrm{Im} E \subset \mathcal{V}^*$$

Proof: If $\mathrm{Im} E \subset \mathcal{V}^*$, then any F such that $(A+BF)\mathcal{V}^* \subset \mathcal{V}^*$ implies $e^{(A+BF)t}\mathcal{V}^* \subset \mathcal{V}^*$ and $C\int_0^t e^{(A+BF)(t-\tau)}Ew(\tau)d\tau = 0$.

Proof continued

Conversely if F is decoupling, then

$$\begin{aligned} C(sI-A-BF)^{-1}E &= 0 \\ Ce^{(A+BF)t}E &= 0 \quad \forall t \\ C(A+BF)^i E &= 0 \quad \forall i \end{aligned}$$

Thus $C\langle A+BF|\mathrm{Im}E\rangle = \{0\}$, so $\langle A+BF|\mathrm{Im}E\rangle$ is an (A,B)-invariant subspace contained in $\ker C$. Therefore $\langle A+BF|\mathrm{Im}E\rangle \subset \mathcal{V}^*$, which is the largest (A,B)-invariant subspace contained in $\ker C$. Hence

$$\mathrm{Im}E \subset \langle A+BF|\mathrm{Im}E\rangle \subset \mathcal{V}^*$$

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Theorem 19.10- Disturbance Decoupling

If (A, B) is controllable and $\mathrm{Im}E \subset \mathcal{R}^*$, then disturbance decoupling can be achieved with arbitrary eigenvalue assignment.

Proof: Use Theorem 1 with $\mathcal{V} = \mathcal{R}^*$. Then $\hat{A}_{22}$ disappears and $(\hat{A}_{11}, \hat{B}_{11})$ is controllable. Moreover $(\hat{A}_{33}, \hat{B}_{32})$ is controllable since (A, B) is controllable. Therefore and F_{11} and F_{23} can be chosen to assign the eigenvalues of

$$A_{11} + B_{11}F_{11} \text{ and } A_{33} + B_{32}F_{23}$$

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Noninteracting Control

Given

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y_j &= C_j x, \quad j = 1, \ldots, m \end{aligned}$$

find a feedback law

$$u = Fx + \sum_i G_i v_i$$

such that the closed loop system is output controllable, but

$$C_j(sI - A - BF)^{-1}BG_i = 0 \text{ for } j \neq i$$

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Geometric Noninteracting Control

Find F and controllability subspaces

$$\mathcal{R}_1, \ldots, \mathcal{R}_m$$

such that

$$\begin{aligned} \mathcal{R}_i &= \langle A + BF | \mathrm{Im}B \cap \mathcal{R}_i \rangle \\ \mathcal{R}_i &\subset \cap_{j \neq i} \mathrm{Ker} C_j \\ C_i \mathcal{R}_i &= \mathrm{Im} C_i \end{aligned}$$

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Theorem 19.11 Noninteracting Control

The noninteracting control problem is solvable if and only if

$$\mathrm{Im}B = \mathrm{Im}B \cap \mathcal{R}_1^* + \cdots + \mathrm{Im}B \cap \mathcal{R}_m^*$$

where $\mathcal{R}_i^*$ is the maximal controllability subspace contained in $\cap_{j \neq i} \mathrm{Ker} C_j$.

Next Lecture

- Norms
- Frequency curves
- Bode gain-phase relation
- Limits of performance