Lecture 4

- Invariant subspace
- Controllable subspace
- Unobservable subspace
- Controlled Invariant Subspace
- Controllability Subspace

The "coordinate free version"

Operations on Linear Subspaces

$$
\begin{aligned}
A &= U\Sigma V^T = \begin{bmatrix} U_{1A} & U_{2A} \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} {V_{1A}}^T \\ {V_{2A}}^T \end{bmatrix} \\
\mathcal{A} &= \mathrm{Im}(A) = \mathrm{Im}(U_{1A}) = \mathrm{Ker}({U_{2A}}^T) \\
\mathcal{B} &= \mathrm{Im}(B) = \\
\mathcal{A} + \mathcal{B} &= \mathrm{Im}([A\,,\,B]) \\
\mathcal{A} \cap \mathcal{B} &= \mathrm{Ker}({U_{2A}}^T) \cap \mathrm{Ker}({U_{2B}}^T) \\
&= \mathrm{Ker}(\begin{bmatrix} U_{2A}^T \\ U_{2B}^T \end{bmatrix}) \\
A\mathcal{B} &= \mathrm{Im}(AB) \\
A^{-1}\mathcal{B} &= \mathrm{Ker}({U_{2B}}^T A)
\end{aligned}
$$

Definition

$\mathcal{V}$ is an A-**invariant subspace** if $A\mathcal{V} \subset \mathcal{V}$

Examples: 0, $\mathrm{Ker}(A)$, $\mathrm{Im}(A)$

Remark: $A\mathcal{V} \subset \mathcal{V} \Rightarrow e^{At}\mathcal{V} \subset \mathcal{V}$

Examples: Eigenvector x of A, $Ax = x\lambda$. Also

$$
A[x_1, x_2] = [x_1, x_2] \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}
$$

Controllable subspace

The minimal A-invariant subspace containing $\mathcal{B}$ is

$$
\langle A|\mathcal{B}\rangle = \mathcal{B} + A\mathcal{B} + \ldots + A^{n-1}\mathcal{B}
$$

$\langle A|\mathcal{B}\rangle$ is also called the **controllable subspace** of

$$
\dot{x} = Ax + Bu
$$

Definition 18.4

A vector x_0 is called a **controllable state** for the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_o \\ y(t) &= Cx(t) \end{aligned}$$

if there is a continuous input

$$u(t), \quad t \in [0, t_a]$$

such that

$$x(t_a) = 0$$

Theorem 18.6+

The following statements are equivalent.

(i) x_0 is a controllable state

(ii) $x_0 \in \mathrm{Im}(W_c(0, t_a))$

(iii) $x_0 \in \langle A \,|\, \mathrm{Im} B \rangle$

Remark: Reachable state more intuitive

Proof that (ii) ⇒ (iii)

$$\begin{aligned} W_c(0, t_a) &= \int_0^{t_a} e^{-A\sigma} B B^T e^{-A^T \sigma} x_0 d\sigma \\ &= \sum_{k=0}^{n-1} A^k B \int_0^{t_a} \alpha_k(-\sigma) B^T e^{-A^T \sigma} x_0 d\sigma \\ &= [B \; AB \ldots A^{n-1}B] \begin{bmatrix} * \\ \\ \vdots \\ * \end{bmatrix} \end{aligned}$$

Proof that (iii) ⇒ (ii)

With P as in Theorem 13.1.

$$P^{-1}AP = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, \quad P^{-1}B = \begin{bmatrix} \hat{B}_{11} \\ 0 \end{bmatrix}$$

$$\hat{W}_c(0, t_a) = \begin{bmatrix} \hat{W}_1 & 0 \\ 0 & 0 \end{bmatrix} = P^{-1} W_c(0, t_a) P^{-T}$$

$$A^k B = P \begin{bmatrix} \hat{A}_{11}^k \hat{B}_{11} \\ 0 \end{bmatrix} = W_c(0, t_a) P^{-T} \begin{bmatrix} \hat{W}_1^{-1} \hat{A}_{11}^k \hat{B}_{11} \\ 0 \end{bmatrix}$$

KTH

Proof that (ii) ⇒ (i)

$$
\begin{aligned}
x_0 &= W_c(0,t_a)z \\
u(t) &= -B^T e^{-A^T t} z \\
x(t_a) &= e^{At_a}x_o - \int_o^{t_a} e^{A(t_a-\sigma)}BB^T e^{-A^T\sigma} z d\sigma \\
&= e^{At_a}[x_o - W_c(0,t_a)z] = 0
\end{aligned}
$$

KTH

Proof that (i) ⇒ (iii)

$$
\begin{aligned}
0 &= e^{At_a}x_o + \int_0^{t_a} e^{A(t_a-\sigma)}Bu_a(\sigma)d\sigma \\
x_o &= -\int_0^{t_a} e^{-A\sigma}Bu_a(\sigma)d\sigma \\
&= \begin{bmatrix} B & AB \cdots A^{n-1}B \end{bmatrix} \begin{bmatrix} * \\ \\ \vdots \end{bmatrix}
\end{aligned}
$$

KTH

Choice of basis

Columns of full rank V basis for $\mathcal{V}$.

Complete V with $\bar{V}$ to basis for $\mathcal{X}$.

$[V, \bar{V}]$ **adapted** to $\mathcal{V}$.

For $\mathcal{V} = \langle A|\mathcal{B}\rangle$

$$
B = \begin{bmatrix} V, & \bar{V} \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}
$$

$$
A \begin{bmatrix} V, & \bar{V} \end{bmatrix} = \begin{bmatrix} V, & \bar{V} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}
$$

A_{11} **restriction** of A to $\mathcal{V}$.

KTH

Unobservable Subspace

The maximal A-invariant subspace contained in $\mathcal{S}$ is

$$
\mathcal{V}_A^*(\mathcal{S}) = \mathcal{S} \cap (A^{-1}\mathcal{S}) \cap (A^{-2}\mathcal{S}) \ldots \cap (A^{-n+1}\mathcal{S})
$$

The subspace

$$
\mathcal{V}_A^*(\mathrm{Ker} C) = \mathrm{Ker}(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix})
$$

is called the **unobservable subspace** for the system

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
$$

Theorem 18.11

The initial state $x(0)$ for the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

can be determined modulo $\mathcal{V}^*(\mathrm{Ker}C)$ from knowledge of $u(t)$ and $y(t)$ for $t \in [0, t_a]$.

Definition

$\mathcal{V}$ is an (A, B)-**invariant subspace** if

$$A\mathcal{V} \subset \mathcal{V} + \mathrm{Im}B$$

Such a subspace is also called a **controlled invariant subspace** of the system

$$\dot{x} = Ax + Bu$$

Theorem 18.19

$$A\mathcal{V} \subset \mathcal{V} + \mathrm{Im}B \quad \Leftrightarrow \quad \exists F: \ (A + BF)\mathcal{V} \subset \mathcal{V}$$

Proof. $\Leftarrow$ is trivial since $BF\mathcal{V} \subset \mathrm{Im}B$.

$\Rightarrow$ Choose a basis $\{v_1, \ldots, v_q\}$ in $\mathcal{V}$. From $A\mathcal{V} \subset \mathcal{V} + \mathrm{Im}B$ we know that $\exists w_i \in \mathcal{V}, u_i \in \mathcal{U}$ such that

$$Av_i = w_i + Bu_i, \quad i = 1, \ldots, q$$

Let F be a feedback matrix such that

$$F[v_1, \ldots, v_n] = -[u_1, \ldots, u_q, 0, \ldots, 0]$$

Then

$$(A + BF)v_i = w_i \in \mathcal{V}, \quad i = 1, \ldots, q$$

Unsolved Problem

The subspaces $\mathcal{V}_1, \ldots, \mathcal{V}_m$ are called **compatible** if there exists a common F such that

$$(A + BF)\mathcal{V}_i \subset \mathcal{V}_i \quad \forall \mathcal{V}_i$$

Necessary and sufficient conditions on a set of subspaces to be compatible are not known.

Loss of Observability by Feedback

Let $\mathcal{V}$ be an (A, B)-invariant subspace of $\mathrm{Ker}C$.

Choose state coordinates $x = [x_1; x_2]$, where x_1 corresponds to a basis for $\mathcal{V}$. Then the feedback law $u = Fx$ gives the system equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & \bar{C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Unobservable Subspace with unknown Inputs

The maximal (A, B)-invariant subspace contained in $\mathcal{S}$ is denoted $\mathcal{V}^*_{(A,B)}(\mathcal{S})$ and can be constructed as follows:

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{S} \\ \mathcal{V}_{i+1} &= \mathcal{S} \cap A^{-1}(\mathcal{V}_i + \mathrm{Im}B) \\ \mathcal{V}^*_{(A,B)}(\mathcal{S}) & \quad \text{is the first } \mathcal{V}_i \text{ such that } \mathcal{V}_i = \mathcal{V}_{i+1} \end{aligned}$$

The initial state $x(0)$ for the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

can be determined modulo $\mathcal{V}^*_{(A,B)}(\mathrm{Ker}C)$ from knowledge of $y(t)$ for $t \in [0, t_a]$.

Controllability Subspaces

$\mathcal{R}$ is **controllability subspace** for the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

if there exist matrices F and G such that

$$\mathcal{R} = \langle A + BF \,|\, \mathrm{Im}(BG) \rangle$$

Remark:

$$\langle A + BF | \mathrm{Im}(BG) \rangle \subset \langle A + BF | \mathrm{Im}B \rangle = \langle A | \mathrm{Im}B \rangle$$

States Reachable with Zero Output

If the controllability subspace $\mathcal{R}$ is a subspace of $\mathrm{Ker}C$, then every state $x \in \mathcal{R}$ can be reached at $t = t_a$ starting from $x(0) = 0$ and maintaining $y(t) = 0$ for $t \in [0, t_a]$.

The sum of all controllability subspaces in $\mathrm{Ker}C$ is called the **maximal controllability subspace in** $\mathrm{Ker}C$, denoted $\mathcal{R}^*$ or $\mathcal{R}^*_{(A,B)}(\mathrm{Ker}C)$.

Calculation of $\mathcal{R}^*_{(A,B)}(\mathrm{Ker}C)$

Introduce the notation $\mathcal{V}^*$ for $\mathcal{V}^*_{(A,B)}(S)$, with $S = \mathrm{Ker}C$.

Let F be such that $(A+BF)\mathcal{V}^* \subset \mathcal{V}^*$.

It actually follows that

$$\mathcal{R}^*_{(A,B)}(S) = \langle A + BF | \mathcal{B} \cap \mathcal{V}^* \rangle$$

and $\mathcal{R}^*$ can be constructed using

$$\begin{aligned} \mathcal{R}_0 &= 0 \\ \mathcal{R}_{i+1} &= \mathcal{V}^* \cap (A\mathcal{R}_i + \mathcal{B}) \\ \mathcal{R}^* & \quad \text{is the first } \mathcal{R}_i \text{ such that } \mathcal{R}_i = \mathcal{R}_{i+1} \end{aligned}$$

State space form

Choose state coordinates $x = (x_1, x_2)$, where x_1 corresponds to a basis for $\mathcal{R}$. Then the feedback law $u = Fx + Gv$ gives the system equation

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ 0 \end{bmatrix} v \\ y &= \begin{bmatrix} 0 & \bar{C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Next lecture

- State Space — Geometric Theory
- Transmission Zeros — Geometric Theory
- Eigenvalue assignment
- Disturbance decoupling
- Noninteracting control