Lecture 3

- Polynomial Matrix Descriptions (PMD's)
- Fuhrmann's System Equivalence
- Decoupling Zeros
- Invariant Zeros

Polynomial Matrix Descriptions

The **system behavior** is the set $\mathcal{B}$ of all (u, y) such that

$$\begin{cases} P\left(\frac{d}{dt}\right)\xi &= Q\left(\frac{d}{dt}\right)u \\ y &= R\left(\frac{d}{dt}\right)\xi + W\left(\frac{d}{dt}\right)u \end{cases}$$

are satisfied for some ξ. (Willems)

$$\begin{bmatrix} P & Q & 0 \\ -R & W & -I \end{bmatrix} w = 0, \quad w = \begin{bmatrix} -\xi \\ u \\ y \end{bmatrix}$$

ξ **latent** variable, $\begin{bmatrix} u \\ y \end{bmatrix}$ **manifest** variable

Examples

$$\begin{cases} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

$$D_L\left(\frac{d}{dt}\right)y = N_L\left(\frac{d}{dt}\right)u$$

$$\begin{cases} D_R\left(\frac{d}{dt}\right)\xi &= u \\ y &= N_R\left(\frac{d}{dt}\right)\xi \end{cases}$$

After Laplace Transformation

$$\begin{cases} P(s)\xi(s) &= Q(s)u(s) \\ y(s) &= R(s)\xi(s) + W(s)u(s) \end{cases}$$

Transfer function (for nonsingular P)

$$H(s) = R(s)P(s)^{-1}Q(s) + W(s)$$

System matrix

$$\mathbf{P}(s) = \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}$$

Transformations on P

$$M_1 P_1 = P_2 M_2$$

can be used to transform P_1 such that P_2 is column or row reduced or transform to some canonical form.

Can also be written

$$\begin{bmatrix} M_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix} = \begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix} \begin{bmatrix} M_2 & 0 \\ 0 & I \end{bmatrix}$$

Remark: M_i are not necessarily square nor unimodular.

Reduction Modulo P

From the division theorem there exist unique Y and Q_2 such that

$$Q_1 = P_1 Y + Q_2$$

where $P_1^{-1} Q_2$ is strictly proper.

In the same way: $\exists\ X$ and R_2: $R_1 = X P_1 + R_2$ and $R_2 P_1^{-1}$ strictly proper.

Application on the system matrix gives

$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix} = \begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix} \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix}$$

where $W_2 = W_1 + X Q_1 - R_2 Y$, and $P_1 = P_2$.

Transfer Function preserving Transformations

The previous transformations can be combined into:

$$\begin{bmatrix} M_1 & 0 \\ X & I \end{bmatrix} \begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix} = \begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix} \begin{bmatrix} M_2 & -Y \\ 0 & I \end{bmatrix}$$

$\mathbf{P}_1(s)$ and $\mathbf{P}_2(s)$ satisfying the above relation have

$$R_1(s) P_1(s)^{-1} Q_1(s) + W_1(s) = R_2(s) P_2(s)^{-1} Q_2(s) + W_2(s)$$

State Space Description from PMD

Consider a PMD with P row reduced.

1) Make $P^{-1}Q$ strictly proper using Y

3) Compute an observer form realization:

$$C_o(sI - A_o)^{-1} = P(s)^{-1} \Psi_L(s)$$
$$C_o(sI - A_o)^{-1} B_o = P(s)^{-1} Q(s)$$

4) Divide $R(s) C_o$ by $sI - A_o$:

$$R(s) C_o = \Lambda(s)(sI - A_o) + C$$

where $C = \hat{R}(A_o) C_o$

Result

Transfer function:

$$R(s)P(s)^{-1}Q(s)+W(s) = C(sI-A_o)^{-1}B_o+\underbrace{W(s) + R(s)Y(s) + \Lambda(s)B_o}_{J(s)}$$

System matrices:

$$\begin{bmatrix} \Psi_L(s) & 0 \\ -\Lambda(s) & I \end{bmatrix}\begin{bmatrix} sI - A_o & B_o \\ -C & J(s) \end{bmatrix} = \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}\begin{bmatrix} C_o & -Y(s) \\ 0 & I \end{bmatrix}$$

$sI - A_o$, C_o are right coprime.

$\Psi_L(s)$, $P(s)$ are left coprime.

Fuhrmann's System Equivalence

$$\begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix} \overset{F}{\sim} \begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix}$$

means existence of M_1, X, M_2, Y such that

$$\begin{bmatrix} M_1 & 0 \\ X & I \end{bmatrix}\begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix} = \begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix}\begin{bmatrix} M_2 & -Y \\ 0 & I \end{bmatrix}$$

M_1, P_2 are left coprime

P_1, M_2 are right coprime

Remark

$$M_1P_1 = P_2M_2, \quad P_2^{-1}M_1 = M_2P_1^{-1}$$

Notice P_1 and P_2 square, invertible, possibly of different size!

Example

$$\begin{bmatrix} s^2 & 1 \\ -1 & 0 \end{bmatrix} \overset{F}{\sim} \begin{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ -\begin{bmatrix} 1 & 0 \end{bmatrix} & 0 \end{bmatrix}$$

Symmetry

$P_2^{-1}M_1 = M_2P_1^{-1}$ both coprime $\Rightarrow \exists\ X^*, Y^*, \tilde{X}, \tilde{Y}$:

$$\begin{bmatrix} -M_2 & X^* \\ P_1 & Y^* \end{bmatrix}\begin{bmatrix} -\tilde{X} & \tilde{Y} \\ P_2 & M_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Then

$$\begin{bmatrix} M_1 & 0 \\ X & I \end{bmatrix}\begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix} = \begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix}\begin{bmatrix} M_2 & -Y \\ 0 & I \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix}\begin{bmatrix} \tilde{X} & \tilde{X}Y - \tilde{Y}Q_1 \\ 0 & I \end{bmatrix} = \begin{bmatrix} Y^* & 0 \\ R_2X^* - XY^* & I \end{bmatrix}\begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix}$$

where P_1, Y^* l. coprime, $\tilde{X}$, P_2 r. coprime. So $\mathbf{P}_1 \overset{F}{\sim} \mathbf{P}_2 \Rightarrow \mathbf{P}_2 \overset{F}{\sim} \mathbf{P}_1$.

Fuhrmann's System Equivalence is an Equivalence Relation

1. Symmetry has already been proven.
2. $\mathbf{P} \overset{F}{\sim} \mathbf{P}$ is obvious.
3. Exercise to prove $\mathbf{P}_1 \overset{F}{\sim} \mathbf{P}_2 \,\&\mathbf{P}_2 \overset{F}{\sim} \mathbf{P}_3 \Rightarrow \mathbf{P}_1 \overset{F}{\sim} \mathbf{P}_3$.

Theorem 1

Two Fuhrmann equivalent system matrices generate identical system behaviors.

Proof: If

$$\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix} \begin{bmatrix} -\xi \\ u \end{bmatrix}$$

then $\mathcal{B}_1 \subset \mathcal{B}_2$, since

$$\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} M_1 & 0 \\ X & I \end{bmatrix} \begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix} \begin{bmatrix} -\xi \\ u \end{bmatrix} = \begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix} \begin{bmatrix} M_2 & -Y \\ 0 & I \end{bmatrix} \begin{bmatrix} -\xi \\ u \end{bmatrix}$$

$$= \begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix} \begin{bmatrix} -M_2\xi - Yu \\ u \end{bmatrix}$$

Theorem 8.2-1

The state-space realizations $(A_i, B_i, C_i, J(s))$ are FSE if and only if they are similar.

Proof: If they are similar it is almost trivial that they are FSE. The other direction is more tricky, see Kailath. However, it will be made very plausible later on, and is therefore omitted.

Theorem 8.2-2

Two right MFDs $N_1(s)D_1^{-1}(s)$ and $N_2(s)D_2^{-1}(s)$ are FSE if and only if there exists a unimodular matrix $U(s)$ such that

$$N_1(s) = N_2(s)U(s); \qquad D_1(s) = D_2(s)U(s)$$

Proof: Assume that the two right MFDs are FSE. Then

$$\begin{bmatrix} M_1 & 0 \\ X & I \end{bmatrix} \begin{bmatrix} D_1 & I \\ -N_1 & 0 \end{bmatrix} = \begin{bmatrix} D_2 & I \\ -N_2 & 0 \end{bmatrix} \begin{bmatrix} M_2 & -Y \\ 0 & I \end{bmatrix}$$

from which it follows that $M_1 = I - D_2Y$ and $M_1D_1 = D_2M_2$. Then

$$D_1 = D_2(M_2 + YD_1) = D_2U$$

Similar it can be shown that $N_1 = N_2U$.

Proof continued

It remains to prove that U is unimodular. The following generalized Bezout identity holds

$$\begin{bmatrix} -\tilde{X} & \tilde{Y} \\ D_2 & M_1 \end{bmatrix} \begin{bmatrix} -M_2 & X^* \\ D_1 & Y^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Since $D_2 Y + M_1 = I$ it follows that $X^* = Y$ and $Y^* = I$. Hence

$$\begin{bmatrix} -M_2 & Y \\ D_1 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -D_1 & I \end{bmatrix} = \begin{bmatrix} -U & Y \\ 0 & I \end{bmatrix}$$

is unimodular proving that U is unimodular.

The converse part is easy.

Extended System Matrix

Consider two FSE system matrices and use the corresponding generalized Bezout identity to write

$$\begin{bmatrix} -\tilde{X} & \tilde{Y} & 0 \\ P_2 & M_1 & 0 \\ -R_2 & X & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & P_1 & Q_1 \\ 0 & -R_1 & W_1 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & P_2 & Q_2 \\ 0 & -R_2 & W_2 \end{bmatrix} \begin{bmatrix} -\tilde{X} & \tilde{Y}P_1 & \tilde{Y}Q_1 \\ I & M_2 & -Y \\ 0 & 0 & I \end{bmatrix}$$

Notice that

$$\begin{bmatrix} I & \tilde{X} \\ 0 & I \end{bmatrix} \begin{bmatrix} -\tilde{X} & \tilde{Y}P_1 \\ I & M_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & M_2 \end{bmatrix}$$

Hence all matrices above are unimodular.

Rosenbrock's System Equivalence

Two PMD's are **Rosenbrock System Equivalent** (RSE) if there exist polynomial matrices X, Y, M_{1e}, M_{2e} such that

$$\begin{bmatrix} M_{1e} & 0 \\ X & I \end{bmatrix} \begin{bmatrix} P_{1e} & Q_{1e} \\ -R_{1e} & W_1 \end{bmatrix} = \begin{bmatrix} P_{2e} & Q_{2e} \\ -R_{2e} & W_2 \end{bmatrix} \begin{bmatrix} M_{2e} & -Y \\ 0 & I \end{bmatrix}$$

where M_{1e} and M_{2e} are unimodular.

Theorem 8.2-4 RSE is equivalent to FSE.

Proof: One direction is already done. For the other one see Kailath.

Definition: If two polynomial matrices $A(s)$ and $B(s)$ have the same non-unity invariant polynomials we write $A(s) \overset{I}{\sim} B(s)$

Theorem 8.2-3

If $\mathbf{P}_1$ and $\mathbf{P}_2$ are FSE then

1. $P_1(s) \overset{I}{\sim} P_2(s)$
2. $\begin{bmatrix} P_1(s) & Q_1(s) \end{bmatrix} \overset{I}{\sim} \begin{bmatrix} P_2(s) & Q_2(s) \end{bmatrix}$
3. $\begin{bmatrix} P_1^T(s) & -R_1^T(s) \end{bmatrix}^T \overset{I}{\sim} \begin{bmatrix} P_2^T(s) & -R_2^T(s) \end{bmatrix}^T$
4. $\mathbf{P}_1(s) \overset{I}{\sim} \mathbf{P}_2(s)$

Proof: Since $M_{1e}P_{1e} = P_{2e}M_{2e}$ it holds that $P_{1e} \sim P_{2e}$. Because of the structure of P_{1e} and P_{2e} it holds that $P_1 \overset{I}{\sim} P_2$. The other claims are proven similarly.

Irreducible PMD

Definition: A PMD $\mathbf{P}$ is said to be **irreducible** if P and Q are left coprime and R and P are right coprime.

Consider the observer-form realization of a PMD, which is of order $n = \deg \det P(s)$. Then

$$\begin{bmatrix} \Psi_L(s) & 0 \\ -\Lambda(s) & I \end{bmatrix} \begin{bmatrix} sI - A_o & B_o \\ -C & J(s) \end{bmatrix} = \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix} \begin{bmatrix} C_o & -Y(s) \\ 0 & I \end{bmatrix}$$

where $sI - A_o$, C_o right coprime and $\Psi_L(s)$, $P(s)$ left coprime. From Theorem 8.2-3 it follows that

P, Q l. coprime $\Leftrightarrow$ $sI - A_o$,B_o l. coprime (**controllability**)

R, P r. coprime $\Leftrightarrow$ C, $sI - A_o$ B_o r. coprime (**observability**)

Theorem 8.3-1

All state-space realizations of a PMD $\mathbf{P}$ of order $n = \deg \det P(s)$ will be minimal if and only if the PMD is irreducible.

Proof: Follows from the results of the previous two slides.

Theorem 8.3-2

All irreducible PMDs of a given transfer function are FSE.

Proof: Since the PMDs are irreducible, the state-space realizations will be minimal by Theorem 8.3-1. All minimal state-space realizations of a given transfer function are similar and therefore FSE. Then by the transitivity property of equivalence relations, the PMDs must be FSE.

Theorem 8.3-3

Given a PMD $\mathbf{P}$ with P and Q left coprime, there exist polynomial matrices N and D such that

$$\begin{bmatrix} P & Q \\ -R & W \end{bmatrix} \overset{F}{\sim} \begin{bmatrix} D & I \\ -N & 0 \end{bmatrix}$$

If $\mathbf{P}$ is irreducible, then N and D are right coprime.

Sketch of Proof: Since P and Q are left coprime, the observer-form realization of the PMD will be controllable. Therefore it is similar to a controller-form realization. Derive a right MFD from this realization.

Poles of an Irreducible PMD

For an irreducible PMD $\mathbf{P}$ the poles of the corresponding transfer function $G(s)$ are given by the roots of $\det P(s) = 0$.

Proof: By Theorem 8.3-3 there is a right coprime MFD $N(s)D^{-1}(s) = G(s)$ which is FSE with $\mathbf{P}$. By Theorem 8.2-3 $D(s) \overset{I}{\sim} P(s)$ which proves the result.

Transmission Zeros of an Irreducible PMD

For an irreducible PMD $\mathbf{P}(s)$ with corresponding transfer function $G(s)$ it holds that s_0 is a transmission zero of $G(s)$ if and only if

$$\operatorname{rank}\mathbf{P}(s_0) < \max_{s\in\mathbf{C}} \operatorname{rank}\mathbf{P}(s)$$

Proof: By Theorem 8.3-3 there is a right coprime MFD $N(s)D^{-1}(s) = G(s)$ which is FSE with $\mathbf{P}$, and moreover

$$\mathbf{P} = \begin{bmatrix} P & Q \\ -R & W \end{bmatrix} \overset{F}{\sim} \begin{bmatrix} D & I \\ -N & 0 \end{bmatrix}$$

From Theorem 8.2-3 it now follows that $N(s)$ drops rank if and only if $\mathbf{P}(s)$ drops rank.

Controllability and Observability

For any state space realization $(A, B, C, J(s))$ that is FSE to a PMD $\mathbf{P}(s)$ it holds by Theorem 8.2-3 that

1. $sI - A \overset{I}{\sim} P(s)$
2. $\begin{bmatrix} sI - A & B \end{bmatrix} \overset{I}{\sim} \begin{bmatrix} P(s) & Q(s) \end{bmatrix}$
3. $\begin{bmatrix} sI - A^T & C^T \end{bmatrix}^T \overset{I}{\sim} \begin{bmatrix} P^T(s) & R^T(s) \end{bmatrix}^T$

Kalman's Decomposition

$$\begin{bmatrix} sI - A & B \\ -C & J(s) \end{bmatrix} \sim \begin{bmatrix} sI - \bar{A} & \bar{B} \\ -\bar{C} & J(s) \end{bmatrix}$$

$$= \begin{bmatrix} sI - A_{co} & 0 & -A_{13} & 0 & B_1 \\ -A_{21} & sI - A_{c\bar{o}} & -A_{23} & -A_{24} & B_2 \\ 0 & 0 & sI - A_{o\bar{c}} & 0 & 0 \\ 0 & 0 & -A_{43} & sI - A_{\bar{o}\bar{c}} & 0 \\ -C_1 & 0 & -\bar{C}_2 & 0 & J(s) \end{bmatrix}$$

Uncontrollable Modes

Because of controllability

$$\begin{bmatrix} sI - A_{co} & 0 & B_1 \\ -A_{21} & sI - A_{c\bar{o}} & B_2 \end{bmatrix} \sim \begin{bmatrix} I & 0 \end{bmatrix}$$

and hence

$$\begin{bmatrix} P(s) & Q(s) \end{bmatrix} \overset{I}{\sim} \begin{bmatrix} sI - A & B \end{bmatrix} \sim \begin{bmatrix} sI - \bar{A} & \bar{B} \\ -\bar{C} & J(s) \end{bmatrix}$$

$$\sim \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & sI - A_{o\bar{c}} & 0 \\ 0 & 0 & -A_{43} & sI - A_{\bar{o}\bar{c}} \end{bmatrix} \overset{I}{\sim} \begin{bmatrix} sI - A_{o\bar{c}} & 0 \\ -A_{43} & sI - A_{\bar{o}\bar{c}} \end{bmatrix}$$

Input Decoupling Zeros

The **input decoupling zeros** of a PMD are defined as the values of s_0 for which

$$\text{rank}\begin{bmatrix} P(s_0) & Q(s_0) \end{bmatrix} < \max_{s\in\mathbf{C}} \text{rank}\begin{bmatrix} P(s) & Q(s) \end{bmatrix}$$

Remark: An irreducible PMD does not have any input decoupling zeros.

Output Decoupling Zeros

The **output decoupling zeros** of a PMD are defined as the values of s_0 for which

$$\text{rank}\begin{bmatrix} P(s_0) \\ R(s_0) \end{bmatrix} < \max_{s\in\mathbf{C}} \text{rank}\begin{bmatrix} P(s) \\ R(s) \end{bmatrix}$$

Remark: An irreducible PMD does not have any output decoupling zeros.

Input-Output Decoupling Zeros

If a PMD $\mathbf{P}$ has input decoupling zeros there is a non-unimodular greatest common left divisor $Z_{id}(s)$ of $P(s)$ and $Q(s)$. Then write

$$P(s) = Z_{id}(s)P_{id}(s); \quad Q(s) = Z_{id}(s)Q_{id}(s)$$

where $P_{id}(s)$ and $Q_{id}(s)$ are left coprime. Then

$$\mathbf{P}(s) = \begin{bmatrix} Z_{id}(s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{id}(s) & Q_{id}(s) \\ -R(s) & W(s) \end{bmatrix}$$

The reduced PMD defined by the rightmost matrix above has no input decoupling zeros. In the reduction process output decoupling zeros corresponding to eigenvalues of $A_{\bar{o}\bar{c}}$ have also been removed. These are called **input-output decoupling zeros**.

Decoupling Zeros and Poles

decoupling zeros = i.d. zeros + o.d. zeros − i.o.d. zeros

poles of a PMD = poles of the transfer function + decoupling zeros

Remark: The interpretation of poles for a PMD for the special case of a state-space realization is that the eigenvalues of the A-matrix are equal to the poles of the transfer function plus the decoupling zeros, where the decoupling zeros are the unobservable and uncontrollable eigenvalues of A.

Invariant Zeros

For a PMD $\mathbf{P}(s)$, not necessarily irreducible, its **invariant** zeros are defined to be the values s_0 for which

$$\operatorname{rank} \mathbf{P}(s_0) < \max_{s \in \mathbf{C}} \operatorname{rank} \mathbf{P}(s)$$

Remark 1: If the PMD is irreducible, the invariant zeros are equal to the transmission zeros.

Remark 2: If the corresponding transfer function of the PMD is invertible, then the invariant zeros are equal to the transmission zeros + the decoupling zeros.

Next Week

- Invariant subspace
- Controllable subspace
- Unobservable subspace
- Controlled invariant subspace
- Controllability subspace