Lecture 2

- MFD $\rightarrow$ Realization
- Realization $\rightarrow$ MFD
- Poles and Zeros

Definition of McMillan Degree

If $N(s)D(s)^{-1}$ is a right coprime MFD of $G(s)$, then the degree of $D(s)$ is called the **McMillan degree** of $G(s)$.

Remark: A left coprime MFD would give the same McMillan degree by Theorem 16.19.

Lemma 17.2

The dimension of any realization of a strictly proper $G(s)$ is at least the McMillan degree of $G(s)$.

Proof: Suppose (A, B, C) is a dimension n minimal realization of $G(s)$. Then

$$\text{rank} \begin{bmatrix} sI - A & B \end{bmatrix} \equiv n$$

so the left polynomial factorization $(sI - A)^{-1}B$ is coprime. Let $N(s)D(s)^{-1}$ be a right coprime factorization of $(sI - A)^{-1}B$. Then $D(s)$ has degree n by Theorem 16.19, and $G(s) = [CN(s)]D(s)^{-1}$, which is not necessarily coprime. For a minimal realization of dimension n there is thus a right MFD of degree n, so a coprime one would have at most degree n. Hence the McMillan degree of $G(s)$ is at most n.

Theorem

The minimal dimension n of a realization of $G(s)$ is equal to the McMillan degree of $G(s)$.

Proof: It is by Lemma 17.2 sufficient to show that any right coprime MFD of $G(s)$ with McMillan degree n has a realization of dimension n. This will be done in what follows.

Notation

$$A_o = \text{block diag}\left\{ \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}_{(\alpha_i \times \alpha_i)}, \; i = 1, \ldots, k \right\}$$

$$\Delta(s) = \text{diag}\left\{ s^{\alpha_1}, \ldots, s^{\alpha_k} \right\}$$

Notation continued

$$B_o = \text{block diag}\left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{(\alpha_i \times 1)}, i = 1, \ldots, k \right\}$$

$$\Psi(s) = \text{block diag}\left\{ \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\alpha_i - 1} \end{bmatrix}, i = 1, \ldots, k \right\}$$

Theorem 17.4

Let $G(s)$ be a strictly proper transfer function with MFD $N(s)D^{-1}(s)$ where $D(s)$ is column reduced and $c_1[D], \ldots, c_m[D] \geq 1$. Then $G(s)$ has the realization

$$\begin{cases} \dot{x}(t) = \overbrace{(A_o - B_o D_{hc}^{-1} D_l)}^{A_c} x(t) + \overbrace{B_o D_{hc}^{-1}}^{B_c} u(t) \\ y(t) = \underbrace{N_l}_{C_c} x(t) \end{cases}$$

where

$$N(s) = N_l \Psi(s)$$
$$D(s) = D_{hc} \Delta(s) + D_l \Psi(s)$$

with $[\alpha_1, \ldots, \alpha_m] = [c_1[D], \ldots, c_m[D]]$.

Remarks

If D_{hc} is upper triangular with unity diagonal elements this is the controller form, and its controllability indices are the column degrees.

If $D_{hc} = I$ the coefficients of the A and C matrices are directly obtained from the coefficients of $D(s)$ and $N(s)$.

If $N(s)$ and $D(s)$ are right coprime, then the realization is minimal.

Example

$$N(s)D^{-1}(s) = \begin{bmatrix} s+1 & 3s \\ s & 1 \end{bmatrix} \begin{bmatrix} (s+1)^2 & 2s \\ 1 & s^2+3 \end{bmatrix}^{-1}$$

$$A_c = \begin{bmatrix} -2 & -1 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Proof

The definition of A_o, B_o, Ψ, Δ gives

$$B_o\Delta(s) = (sI - A_o)\Psi(s)$$

Hence

$$\begin{aligned} B_o D_{hc}^{-1}[D(s) - D_l\Psi(s)] &= (sI - A_o)\Psi(s) \\ [sI - A_o + B_o D_{hc}^{-1} D_l]^{-1} B_o D_{hc}^{-1} &= \Psi(s)D(s)^{-1} \\ N_l[sI - A_o + B_o D_{hc}^{-1} D_l]^{-1} B_o D_{hc}^{-1} &= N(s)D(s)^{-1} \end{aligned}$$

If $N(s)$ and $D(s)$ are right coprime, then by Lemma 17.2 this realization is minimal since it has dimension equal to the McMillan degree.

Kailath: Lemma 6.4-1

The realization (A_c, B_c, C_c) of the column-reduced (not necessarily coprime) right MFD in Theorem 17.4 $N(s)D^{-1}(s)$ can be related as

$$\begin{bmatrix} sI - A_c & B_c \\ -C_c & 0 \end{bmatrix} \begin{bmatrix} \Psi(s) & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} B_c & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D(s) & I \\ -N(s) & 0 \end{bmatrix}$$

where $\Psi(s)$ and $D(s)$ are r. coprime and $sI - A_c$ and B_c are l. coprime.

Proof: From the proof of Theorem 17.4 $(sI - A_c)\Psi(s) = B_c D(s)$ and $C_c\Psi(s) = N(s)$, from which the matrix formula follows. (A_c, B_c) is controllable, since obtained by input transformation and feedback around (A_o, B_o) which is controllable. Therefore $sI - A_c$ and B_c l. coprime by the PBH-test. Since $\Psi(s)$ has rank m for all s, $\Psi(s)$ and $D(s)$ are r. coprime by Theorem 16.9.

Generalized Bezout Identity

Notice that $\Psi(s)D^{-1}(s) = (sI - A_c)^{-1}B_c$, where both MFD's are coprime. Hence by applying the ideas in the proof of Theorem 16.19 it is not difficult to verify that there exist polynomial matrices $\tilde{X}(s)$, $\tilde{Y}(s)$, $X(s)$, and $Y(s)$ such that

$$\begin{bmatrix} sI - A_c & B_c \\ -\tilde{X}(s) & \tilde{Y}(s) \end{bmatrix} \begin{bmatrix} X(s) & -\Psi(s) \\ Y(s) & D(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Remark: The matrix multiplication in the above identity can also be done in the reversed order.

Consequence

$$\begin{bmatrix} sI - A_c & B_c \\ 0 & I \end{bmatrix} \begin{bmatrix} X(s) & -\Psi(s) \\ Y(s) & D(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ Y(s) & D(s) \end{bmatrix}$$

Since the second block matrix is unimodular it follows by elementary operations that $sI - A_c$ and $D(s)$ have the same invariant polynomials.

Definition: A **pole** of the transfer matrix $G(s)$ with right coprime factorization $N(s)D(s)^{-1}$ is a complex number s_0 such that

$$\det D(s_0) = 0$$

Lemma

Let (A, B, C) be a minimal realization of the strictly proper transfer matrix $G(s)$. Then s_0 is a pole of $G(s)$ if and only if s_0 is an eigenvalue of A.

Proof: Let $N(s)D(s)^{-1}$ be a right coprime MFD of $G(s)$, and let (A_c, B_c, C_c) be the corresponding minimal controller realization. Then A_c must have the same eigenvalues as A. Since $sI - A_c$ and $D(s)$ have the same invariant polynomials, the result follows.

Theorem 17.9+

Let $G(s)$ have the minimal realization (A, B, C). Then the following statements are equivalent.

(i) s_0 is a pole of $G(s)$

(ii) s_0 is an eigenvalue of A

(iii) $\exists x_0, y_0 : \; Ce^{At}x_0 = y_0 e^{s_0 t} \neq 0, \quad t \geq 0$

Proof: (i)⇔(ii) was proven in the previous Lemma.

(ii)⇒(iii): Let x_0 be the eigenvector corresponding to s_0

(iii)⇒(ii): After Laplace transformation

$$\begin{aligned} C(sI - A)^{-1}x_0 &= y_0(s - s_0)^{-1} \\ (s - s_0)C\text{adj}(sI - A)x_0 &= y_0 \det(sI - A) \end{aligned}$$

Definition

Let $G(s)$ have the right coprime factorization $N(s)D(s)^{-1}$. Then s_0 is called a **transmission zero** of $G(s)$ if

$$\text{rank} N(s_0) < \max_{s \in \mathbf{C}} \text{rank}\, N(s)$$

i.e. the rank of $N(s)$ drops below its normal rank for $s = s_0$.

Example

$$\begin{bmatrix} \frac{s+2}{(s+1)^2} & 0 \\ 0 & \frac{s+1}{(s+2)^2} \end{bmatrix} =$$

$$\begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} (s+1)^2 & 0 \\ 0 & (s+2)^2 \end{bmatrix}^{-1}$$

has two zeros $-1, -2$
and two double poles $-1, -2$.

It is straight-forward to determine the zeros if $N(s)$ is on Smith-form or Hermite-form.

Lemma

Let (A, B, C) be a minimal realization of $G(s)$. Then s_0 is a transmission zero of $G(s)$ if and only if

$$\text{rank} \begin{bmatrix} s_0 I - A & B \\ -C & 0 \end{bmatrix} < \max_{s \in \mathbf{C}} \text{rank} \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}$$

Proof: Let $N(s)D(s)^{-1}$ be a right coprime MFD of $G(s)$, and let (A_c, B_c, C_c) be the corresponding minimal controller realization. Then there is a state transformation T such that

$$\begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix} \sim \begin{bmatrix} sI - A_c & B_c \\ -C_c & 0 \end{bmatrix}$$

Proof continued

From the generalized Bezout identity it follows that

$$\begin{bmatrix} sI - A_c & B_c \\ -C_c & 0 \end{bmatrix} \begin{bmatrix} X(s) & -\Psi(s) \\ Y(s) & D(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -C_c X(s) & N(s) \end{bmatrix}$$

where the second block matrix is unimodular. Hence the rank of $N(s)$ drops below its normal rank if and only if the rank of the leftmost block-matrix drops below its normal rank.

Theorem 17.13 modified

Let $G(s)$ have the minimal realization (A, B, C), where B is $n \times m$. Then the following conditions are equivalent.

(i) $\text{rank} \begin{bmatrix} s_0 I - A & B \\ -C & 0 \end{bmatrix} < n + m$

(ii) $\exists u_0 \neq 0, x_0 : \; Ce^{At}x_0 + \int_0^t Ce^{A(t-\sigma)} B u_0 e^{s_0 \sigma} d\sigma \equiv 0$

Remark 1: The vector u_0 is sometimes called input zero-direction.

Remark 2: If s_0 is a transmission zero of $G(s)$ then (i) holds. The converse is not true in general.

Proof

(i)⇒(ii): Then there is $\begin{bmatrix} x_0^T & u_0^T \end{bmatrix}^T \neq 0$ such that

$$\begin{bmatrix} s_0 I - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} -x_0 \\ u_0 \end{bmatrix} = 0$$

Moreover $u_0 \neq 0$, since otherwise (C, A) would not be observable. Let $u(t) = u_0 e^{s_0 t}$ with Laplace transform $U(s) = u_0/(s - s_0)$. Then

$$\mathcal{L}\left\{ Ce^{At}x_0 + \int_0^t Ce^{A(t-\sigma)} B u_0 e^{s_0 \sigma} d\sigma \right\} = C(sI - A)^{-1} \left[x_0 + B \frac{u_0}{s - s_0} \right]$$
$$= \frac{1}{s - s_0} C(sI - A)^{-1} \left[(s - s_0)x_0 + (s_0 I - A)x_0 \right] = \frac{1}{s - s_0} Cx_0 = 0$$

Proof continued

(ii)⇒(i): There is s_0, x_0 and $u_0 \neq 0$ such that

$$CX(s) = 0$$
$$(sI - A)X(s) = x_0 + B \frac{u_0}{s - s_0}$$

for some $X(s)$. The goal is to find $\begin{bmatrix} \xi_1^T & \xi_2^T \end{bmatrix}^T \neq 0$ such that

$$\begin{bmatrix} s_0 I - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$$

Case 1: Assume that $s = s_0$ is not a pole of $X(s)$. Then

$$(sI - A)X(s)(s - s_0) - x_0(s - s_0) + Bu_0$$

and using $s = s_0$ we get $Bu_0 = 0$. Hence take $\xi_1 = 0, \xi_2 = u_0$.

Proof continued

Case 2: Assume that $s = s_0$ is a pole of $X(s)$ with multiplicity n. Then $X(s) = \bar{X}(s)/(s - s_0)^n$ where $\bar{X}(s_0) \neq 0$, and

$$(sI - A)\bar{X}(s) = x_0(s - s_0)^n + B_c u_0 (s - s_0)^{n-1}$$
$$C\bar{X}(s) = 0$$

Let $s = s_0$. In case $n > 1$ it follows that (C, A) is not observable, and hence $n = 1$. Then use $\xi_1 = -\bar{X}(s_0), \xi_2 = u_0$.

Example

$$\begin{bmatrix} s+2 & 1 & 1 \\ -1 & s & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

The rank drops from 3 to 2 for $s = -3$.

The transfer function is $G(s) = \frac{s+3}{(s+1)^2}$.

Poles and Zeros at infinity

For

$$G(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

it holds that

$$G(1/\lambda) = \begin{bmatrix} 1 & 1/\lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}^{-1}$$

Hence $G(s)$ has one zero and one pole at infinity.

Next week

- PMD's
- More on zeros
- Fractions of stable matrices
- Normalized Coprime Factorizations
- Parameterization of Stabilizing Controllers