## Linear Systems II

- Matrix Fraction Descriptions (MFD's)
- Poles and Zeros
- MFD versus State Space Description
- Other System Descriptions
- Geometrical Theory
- Disturbance Decoupling
- Performance/Robustness Measures

Rugh, Linear System Theory, chapter 16-19.

Additional material.



## Lecture 1

- Hermite form
- Matrix Fraction Descriptions (MFD's)
- Column/Row Degree



## Definitions

**Polynomial Matrix** matrix with polynomial entries

**Nonsingular** square and $\det P(s)$ is not $\equiv 0$

**Unimodular** $\det P(s)$ is a nonzero constant

**Right Polynomial Fraction** $N(s)D(s)^{-1}$

**Left Polynomial Fraction** $D_L(s)^{-1}N_L(s)$

**Degree of** $P(s)$ means $\deg(\det P(s))$



## Example

$$
\begin{aligned}
G(s) &= \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{s}{s+2} & \frac{s+1}{s+2} \end{bmatrix} \\
&= [(s+1)(s+2)I]^{-1} \begin{bmatrix} s+2 & 0 \\ s(s+1) & (s+1)^2 \end{bmatrix} \\
&= \begin{bmatrix} s+2 & 0 \\ s(s+1) & (s+1)^2 \end{bmatrix} [(s+1)(s+2)I]^{-1}
\end{aligned}
$$

## Common Factors

**Right Divisor** $P(s) = \tilde{P}(s)R(s)$

**Common Right Divisor**

$P(s) = \tilde{P}(s)R(s)$, $Q(s) = \tilde{Q}(s)R(s)$

**Greatest Common Right Divisor** All other c.r.d.'s are right divisors of the g.c.r.d.

**Right Coprime** All c.r.d.'s are unimodular



## Example

The polynomial matrices

$$P(s) = \begin{bmatrix} (s+1)^2(s+2) & (s+3)(s+5) \\ 0 & (s+4)(s+5) \end{bmatrix}$$
$$Q(s) = \begin{bmatrix} s+3 & 0 \\ s+5 & s+5 \end{bmatrix}$$

have the common right divisor

$$\begin{bmatrix} 1 & 0 \\ 0 & s+5 \end{bmatrix}$$



## Theorem 16.6

Given

$P(s)$ a $p \times r$ polynomial matrix

$Q(s)$ a $r \times r$ polynomial matrix

if there exist

$U(s)$ a $(p+r) \times (p+r)$ unimodular polynomial matrix

$R(s)$ a $r \times r$ polynomial matrix

such that

$$U(s) \begin{bmatrix} Q(s) \\ P(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}$$

then $R$ is a greatest common right divisor of $P$ and $Q$.



## Proof

With notation

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1}$$

we have

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} V_{11}R \\ V_{21}R \end{bmatrix}$$

so $R$ is a c.r.d. of $P$ and $Q$.

## Proof continued

Moreover, if

$$\begin{aligned} P &= \tilde{P}\tilde{R} \\ Q &= \tilde{Q}\tilde{R} \end{aligned}$$

then

$$\begin{aligned} R &= U_{11}Q + U_{12}P \\ &= (U_{11}\tilde{Q} + U_{12}\tilde{P})\tilde{R} \end{aligned}$$

so $\tilde{R}$ is a right divisor of $R$. Hence $R$ is a g.c.r.d. of $P$ and $Q$.



## Elementary Row Operations

Multiplication from the left by elementary matrices

$$E_a = \begin{bmatrix} I & & & & \\ & 0 & & 1 & \\ & & I & & \\ & 1 & & 0 & \\ & & & & I \end{bmatrix} ; \quad E_c = \begin{bmatrix} I & & & & \\ & 1 & & p(s) & \\ & & I & & \\ & & & 1 & \\ & & & & I \end{bmatrix}$$

$$E_b = \begin{bmatrix} I & & \\ & \alpha & \\ & & I \end{bmatrix}, \quad \alpha \neq 0$$



## Normal Rank

A polynomial matrix $P(s)$ is said to have **normal rank** $r$ if

$$\operatorname{rank} P(s) = r$$

for almost all complex $s$.

**Lemma:** A square polynomial matrix $P(s)$ is nonsingular if and only if it has full normal rank.



## Theorem 16.7+: Row Hermite Form

A $p \times m$ polynomial matrix $P(s)$ of normal rank $r$ can be reduced by elementary row operations to an upper quasi-triangular form in which

1. If $p > r$, the last $p - r$ rows are identically zero;
2. In column $j$, $1 \leq j \leq r$, the diagonal element is monic and of higher degree than any (nonzero) element above it;
3. In column $j$, $1 \leq j \leq r$, if the diagonal element is unity, then all elements above it are zero; and
4. If $m > r$, no particular statement can be made about the elements in the last $m - r$ columns and first $r$ rows.

**Proof:** Similar but easier than the proof of the Smith form.

**Remark:** Unique Hermite form if $P(s)$ nonsingular, see Kailath.

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## Theorem 16.9

For a $p \times r$ polynomial matrix $P(s)$ and a nonsingular $r \times r$ polynomial matrix $Q(s)$, the following statements are equivalent.

**(i)** $P$ and $Q$ are coprime

**(ii)** There exist polynomial matrices $X$ and $Y$ of dimension $p \times r$ and $r \times r$ such that

$$X(s)P(s) + Y(s)Q(s) = I \quad (\textbf{Bezout identity})$$

**(iii)** For every complex number $s$

$$\text{rank} \begin{bmatrix} Q(s) \\ P(s) \end{bmatrix} = r$$



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## Proof

(i)⇒(ii) Reduce to row Hermite form

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix}$$

and let $X = U_{11}$ and $Y = U_{12}$.

(ii)⇒(iii) For all $s \in \mathbf{C}$

$$\begin{aligned} r &= \text{rank}(X(s)P(s) + Y(s)Q(s)) \\ &= \text{rank} \left\{ \begin{bmatrix} Y(s) & X(s) \end{bmatrix} \begin{bmatrix} Q(s) \\ P(s) \end{bmatrix} \right\} \leq \text{rank} \begin{bmatrix} Q(s) \\ P(s) \end{bmatrix} \leq r \end{aligned}$$



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## Proof that (iii)⇒(i)

Let $R$ be a c.r.d. of $P$ and $Q$, such that $P = \tilde{P}R$, $Q = \tilde{Q}R$. Then

$$\begin{aligned} r &= \text{rank} \begin{bmatrix} Q(s) \\ P(s) \end{bmatrix} \\ &= \text{rank} \left\{ \begin{bmatrix} \tilde{Q}(s) \\ \tilde{P}(s) \end{bmatrix} R(s) \right\} \\ &\leq \text{rank}\, R(s) \leq r \end{aligned}$$

so $\text{rank}\, R(s) \equiv r$ and $R$ is unimodular.



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## Theorem 16.10

Given two coprime right polynomial fractions

$$N(s)D(s)^{-1} = N_a(s)D_a(s)^{-1}$$

there exists a unimodular $U(s)$ such that

$$\begin{aligned} N(s) &= N_a(s)U(s) \\ D(s) &= D_a(s)U(s) \end{aligned}$$

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## Proof

By Theorem 16.9, there exist $X$, $Y$ such that

$$\begin{aligned} I &= XN_a + YD_a \\ {D_a}^{-1}D &= XN_a{D_a}^{-1}D + YD \\ &= XN + YD \end{aligned}$$

and $A$, $B$ such that

$$\begin{aligned} I &= AN + BD \\ D^{-1}D_a &= AN_a + BD_a \end{aligned}$$

Both ${D_a}^{-1}D$ and its inverse are polynomial matrices, so they must be unimodular. The choice $U(s) = D_a(s)^{-1}D(s)$ can therefore be used.



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## From Right to Left

Transposing all matrices in the previous results, give corresponding results for left polynomial fractions.



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## Theorem 16.19

Given two coprime polynomial fractions, one right and one left

$$N(s)D(s)^{-1} \quad = \quad D_L(s)^{-1}N_L(s)$$

there exists a nonzero constant $\alpha$ such that

$$\det D(s) \quad = \quad \alpha \det D_L(s)$$

**Remark:** The proof contains a way to construct $D_L(s), N_L(s)$ from $D(s), N(s)$ using the unimodular $U(s)$ that transforms $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ to row Hermite form $\begin{bmatrix} I \\ 0 \end{bmatrix}$.



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## Proof

Introduce $U_{ij}$ and $V_{ij}$ such that

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} D & V_{12} \\ N & V_{22} \end{bmatrix} = I \quad (\textbf{Generalized Bezout identity})$$

$D$ is nonsingular and the same follows for $U_{22} = [V_{22} - ND^{-1}V_{12}]^{-1}$.

$$U_{21}D + U_{22}N = 0; \qquad {U_{22}}^{-1}U_{21} = -ND^{-1} = -{D_L}^{-1}N_L$$

$U_{22}$ and $U_{21}$ are left coprime because $U_{21}V_{12} + U_{22}V_{22} = I$. Hence

$$\begin{aligned} \frac{\det D(s)}{\det D_L(s)} &= \text{const} \cdot \frac{\det D(s)}{\det U_{22}} = \text{const} \cdot \det D(s) \det[V_{22} - ND^{-1}V_{12}] \\ &= \text{const} \cdot \det \begin{bmatrix} D & V_{12} \\ N & V_{22} \end{bmatrix} = \text{const} \end{aligned}$$

## Definition of Column Degree

For a polynomial matrix $P(s)$, the **$j$'th column degree of $P(s)$**, denoted $c_j[P]$, is defined as

$$c_j[P] \quad = \quad \max_i \deg[P_{ij}]$$

A square matrix $P$ is called **column reduced** if

$$\deg \det P(s) \quad = \quad \sum_j c_j[P]$$

Define the **column degree coefficient matrix** $P_{hc}$ by

$$(P_{hc})_{ij} \quad = \quad \lim_{|s|\to\infty} \frac{P_{ij}(s)}{s^{c_j[P]}}$$



## Theorem 16.22

$P(s)$ is column reduced if and only if $P_{hc}$ is invertible

## Theorem 16.24

If $ND^{-1}$ is strictly proper, then $c_j[N] < c_j[D]$.

If $D(s)$ is column reduced and $c_j[N] < c_j[D]$, then $ND^{-1}$ is strictly proper.



## Division Theorem for Polynomial Matrices

Let $D(s)$ be an $m \times m$ nonsingular polynomial matrix. Then, for any $p \times m$ polynomial matrix $N(s)$, there exist unique polynomial matrices $Q(s)$ and $R(s)$ such that

$$N(s) = Q(s)D(s) + R(s)$$

where $R(s)D^{-1}(s)$ is strictly proper

**Proof:** Write $H(s) = N(s)D^{-1}(s) = H_{sp}(s) + P(s)$, where $P(s)$ is polynomial and $H_{sp}(s)$ is strictly proper. Then

$$H(s)D(s) = N(s) = P(s)D(s) + R(s)$$

where $R(s) = H_{sp}(s)D(s)$ is polynomial and $R(s)D^{-1}(s) = H_{sp}(s)$ is strictly proper.



## Proof of Uniqueness

Assume that

$$Q(s)D(s) + R(s) = \tilde{Q}(s)D(s) + \tilde{R}(s)$$

Then

$$Q(s) - \tilde{Q}(s) = [\tilde{R}(s) - R(s)]D^{-1}(s)$$

where the left hand side is polynomial and the right hand side is strictly proper proving that both sides must be zero, from which uniqueness follows.

## Corollary: Division by $sI - A$

For a polynomial matrix

$$F(s) = F_0 s^n + F_1 s^{n-1} + \cdots F_n$$

it holds that

$$Q(s) = F_0 s^{n-1} + (F_0 A + F_1) s^{n-2} + \cdots (F_0 A^{n-1} + \cdots + F_{n-1})$$

and $R(s) = \hat{F}(A)$ are the unique polynomial matrices that satisfy

$$F(s) = Q(s)(sI - A) + R(s)$$

such that $R(s)(sI - A)^{-1}$ is strictly proper



## Proof

Trivially $Q(s)$ and $R(s)$ satisfies $F(s) = Q(s)(sI - A) + R$. Since $D(s) = sI - A$ is column reduced and since $c_j[R] < c_j[D]$ it follows by Theorem 16.24 that $R(s)(sI - A)^{-1}$ is strictly proper. Therfore by the division theorem $Q(s)$ and $R(s)$ are unique.



## Next week

- Realizations/MFD's
- Smith-McMillan Form
- Poles and Zeros