Lecture 7

- Polynomial matrices and Smith normal form
- Differential Algebraic Equations and Pencils
- Weierstrass normal form

Example

General Robot Model

$$J\ddot{x}(t) + D\dot{x}(t) + Kx(t) = f(t)$$

where J, D and K are matrices

Often good to use physical variables and "natural" equations

Interconnection of subsystems

How can a system of linear differential equations on the above form be transformed, and what is the most simple form?

Permissible Transformations

1. On **equations**
   (a) interchange order
   (b) addition of one equation or derivative of this equation to another equation
   (c) multiplication with a constant $c \neq 0$.
2. On **variables**
   (a) renumbering
   (b) replace a variable x_i with $x_i + cx_j$ or $x_i + c\frac{dx_j}{dt}$, $i \neq j$ etc.
   (c) multiplication of x_i with a constant $c \neq 0$.

Algebraic Formulation

$$A\left(\frac{d}{dt}\right)x(t) = f(t)$$

where $A(\lambda) = A_0 + A_1\lambda + \cdots + A_p\lambda^p$ is a matrix polynomial of degree p if $A_p \neq 0$.

For robot example: $A(\lambda) = K + D\lambda + J\lambda^2$.

Elementary Operations

For **rows**:

1. interchange two rows
2. addition of $p(\lambda)$ times row j to row i, $i \neq j$.
3. multiplication of a row with a **constant** $c \neq 0$.

Can be interpreted as multiplying from left with a product of **elementary matrices**.

A sequence of elementary row operations is invertible.

Elementary column operations are defined similarly.

Definition 1: Equivalence

Two polynomial matrices $A(\lambda)$ and $B(\lambda)$ are equivalent if $A(\lambda)$ can be transformed into $B(\lambda)$ using elementary row and column operations. We then write

$$A\lambda) \sim B(\lambda)$$

Remark: $A\lambda) \sim B(\lambda)$ if and only if there exist $P(\lambda)$ and $Q(\lambda)$ such that $B(\lambda) = P(\lambda)A\lambda)Q(\lambda)$ where $P(\lambda)$ and $Q(\lambda)$ are products of elementary matrices.

Some more Definitions

Definition 2: $A(\lambda)$ is **unimodular** if $\det A(\lambda) = c \neq 0$

Definition 3: $A(\lambda)$ is **invertible** if there is $B(\lambda)$ such that

$$A(\lambda)B(\lambda) = B(\lambda)A(\lambda) = I$$

Theorem 1: Invertability

$A(\lambda)$ is invertible if and only if $A(\lambda)$ is unimodular.

Proof: If $A(\lambda)$ is invertible, then there is $B(\lambda)$ such that $A(\lambda)B(\lambda) = I$. Hence $\det A(\lambda) \cdot \det B(\lambda) = 1$ and both $A(\lambda)$ and $B(\lambda)$ are unimodular.

If $A(\lambda)$ is unimodular then

$$A(\lambda)\text{adj}A(\lambda) = \det A(\lambda)I = cI \neq 0$$

and hence $A^{-1}(\lambda) = \text{adj}A(\lambda)/c$ which is a polynomial matrix.

Corollary 1: Products of elementary matrices are unimodular.

Theorem 2: Smith Normal Form

For any polynomial matrix $A(\lambda)$ it holds that

$$A(\lambda) \sim \begin{bmatrix} D_r(\lambda) & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$D_r(\lambda) = \mathrm{diag}(i_1(\lambda), i_2(\lambda), \dots, i_r(\lambda))$$

and where $i_k(\lambda)$ are monic polynomials for which i_k divides i_{k+1} for $k = 1, 2, \dots, r-1$.

Proof

Perform elementary column and row operations so that

$$A(\lambda) \sim \begin{bmatrix} \bar{a}_{11}(\lambda) & \bar{a}_{12}(\lambda) & \cdots & \bar{a}_{1n}(\lambda) \\ \bar{a}_{21}(\lambda) & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{m1}(\lambda) & * & \cdots & * \end{bmatrix}$$

where $\bar{a}_{11}(\lambda) \neq 0$, $\deg \bar{a}_{11}(\lambda) \leq \deg \bar{a}_{i1}(\lambda)$, $i = 2, 3, \dots, m$ and $\deg \bar{a}_{11}(\lambda) \leq \deg \bar{a}_{1j}(\lambda)$, $j = 2, 3, \dots, n$.

Proof continued

Write

$$\bar{a}_{i1}(\lambda) = \bar{a}_{11}(\lambda) q_{i1}(\lambda) + r_{i1}(\lambda)$$
$$\bar{a}_{1j}(\lambda) = \bar{a}_{11}(\lambda) q_{1j}(\lambda) + r_{1j}(\lambda)$$

and make elementary operations so that

$$A(\lambda) \sim \begin{bmatrix} \bar{a}_{11}(\lambda) & r_{12}(\lambda) & \cdots & r_{1n}(\lambda) \\ r_{21}(\lambda) & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1}(\lambda) & * & \cdots & * \end{bmatrix}$$

In case not all $r_{ij}(\lambda) = 0$ start all over.

Proof continued

Now

$$A(\lambda) \sim \begin{bmatrix} \tilde{a}_{11}(\lambda) & 0 & \cdots & 0 \\ 0 & \tilde{a}_{22}(\lambda) & \cdots & \tilde{a}_{2n}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \tilde{a}_{2m}(\lambda) & \cdots & \tilde{a}_{mn}(\lambda) \end{bmatrix}$$

In case $\tilde{a}_{11}(\lambda)$ not divides all $\tilde{a}_{ij}(\lambda)$, $i = 2, 3, \dots, m$, $j = 2, 3, \dots, n$, then add a column containing such an element to the first column and start all over.

Proof continued

Now

$$A(\lambda) \sim \begin{bmatrix} \hat{a}_{11}(\lambda) & 0 & \cdots & 0 \\ 0 & \hat{a}_{22}(\lambda) & \cdots & \hat{a}_{2n}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \hat{a}_{2m}(\lambda) & \cdots & \hat{a}_{mn}(\lambda) \end{bmatrix}$$

where $\hat{a}_{11}(\lambda)$ divides all $\hat{a}_{ij}(\lambda)$, $i = 2, 3, \ldots, m$, $j = 2, 3, \ldots, n$. Repeat the whole procedure on the 2,2-block matrix until it is zero or of zero dimension.

Normalize so that the polynomials are monic.

Definition 4: Determinantal divisors

A **determinantal divisor** $d_j(\lambda)$ of a polynomial matrix $A(\lambda)$ is the greatest common divisor of all the minors of order j in $A(\lambda)$, $j = 1, 2, \ldots, \min(m, n)$.

Lemma 1

The determinantal divisors are invariant under elementary operations.

Proof: Let $B(\lambda) = P(\lambda)A(\lambda)$ where $P(\lambda)$ is a product of elementary matrices. By the Cauchy-Binet formula for determinants

$$\det(B[I,J](\lambda)) = \sum_{\#K=j} \det(P[I,K](\lambda))\det(A[K,J](\lambda))$$

where $\#I = \#J = j$. Since $\det(P[I,K](\lambda)) = c \neq 0$ it follows that $A(\lambda)$ and $B(\lambda)$ have the same determinantal divisors.

Theorem 3

The Smith form is unique

Proof: Any matrix

$$\begin{bmatrix} i_1(\lambda) & 0 & \cdots & 0 & \cdots & 0 \\ 0 & i_2(\lambda) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & & i_r(\lambda) & & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

where i_k divides i_{k+1} for $k = 1, 2, \ldots, r-1$ has

Proof continued

$$d_m(\lambda) = i_1(\lambda)i_2(\lambda)\cdots i_m(\lambda),\ m = 1, 2, \ldots, r$$
$$d_m(\lambda) = 0,\ m > r$$

Hence

$$i_1(\lambda) = d_1(\lambda)$$
$$i_m(\lambda) = d_m(\lambda)/d_{m-1}(\lambda),\ 2 \leq m \leq r$$

Since the determinantal divisors by Lemma 1 are invariant under elementary operations, $i_k(\lambda)$ are uniquely determined by the original matrix.

Definition 5: $i_k(\lambda)$, $k = 1, 2, \ldots, r$ are called the **invariant polynomials** of $A(\lambda)$.

Theorem 4

Two polynomial matrices of the same order are equivalent if and only if they have the same invariant polynomials

Proof: Use elementary operations to bring both matrices to their Smith form. The result follows from the uniqueness of the Smith form by Theorem 3.

Theorem 5

1. A polynomial matrix $A(\lambda)$ is invertible if and only if it is a product of elementary matrices.
2. Two polynomial matrices $A(\lambda)$ and $B(\lambda)$ are equivalent if and only if there are invertible matrices $P(\lambda)$ and $Q(\lambda)$ such that $B(\lambda) = P(\lambda)A(\lambda)Q(\lambda)$

Proof

1. By Theorem 1 $A(\lambda)$ is invertible if and only if it is unimodular. This holds if and only if the Smith form is unimodular, i.e. equal to the identity, but then $A(\lambda)$ is the product of elementary matrices. Also the product of elementary matrices is unimodular by Corollary 1.
2. Follows from 1) and Definition 1.

Application to System of Differential Equations

Given a polynomial matrix $A(\lambda)$ and the differential equation

$$A\left(\frac{d}{dt}\right)x(t) = f(t)$$

Let $P(\lambda)$ and $Q(\lambda)$ such that $B(\lambda) = P(\lambda)A(\lambda)Q(\lambda)$, where $B(\lambda)$ is on Smith form. Then with

$$x(t) = Q\left(\frac{d}{dt}\right)z(t); \quad g(t) = P\left(\frac{d}{dt}\right)u(t)$$

it holds that

$$B\left(\frac{d}{dt}\right)z(t) = g(t)$$

which can be solved by integration, but to compute $g(t)$ and $x(t)$ **derivation** is needed.

Differential Algebraic Equation

Models of physical systems can often be written in a natural way as

$$0 = F(\dot{x}, x, t)$$

An important special case is the linear **Differential Algebraic Equation (DAE)**

$$E\dot{x} = Ax + f(t)$$

Any linear differential equation with higher order derivatives can be brought into this form by augmenting the state vector.

What is the most simple form for a linear DAE?

Definition 6: Associated to the linear DAE is the matrix polynomial $\lambda E - A$ which is called a **matrix pencil**

Example: Two Tank System

Flow: q, Volumes: V_1, V_2, Concentrations: $u(t)$, $x_1(t)$, $x_2(t)$

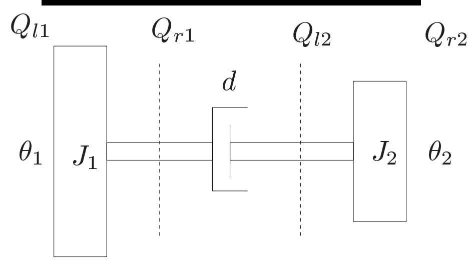
Dynamics:

$$\begin{cases} V_1\dot{x}_1 + qx_1 = qu \\ V_2\dot{x}_2 - qx_1 + qx_2 = 0 \end{cases}$$

$$\dot{x} = \begin{bmatrix} -\frac{1}{V_1} & 0 \\ \frac{1}{V_2} & -\frac{1}{V_2} \end{bmatrix} qx + \begin{bmatrix} \frac{1}{V_1} \\ 0 \end{bmatrix} qu$$

$V_1 = 0$ or $V_2 = 0$?

Example: Rotating Masses



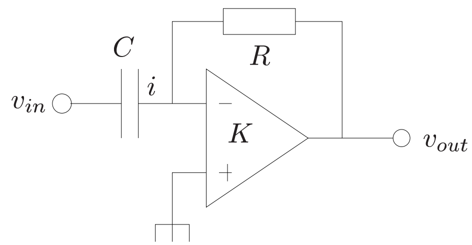
$$J_1\dot{\omega}_1 = Q_{l1} + Q_{r1} \qquad \dot{\theta}_1 = \omega_1$$

$$J_2\dot{\omega}_2 = Q_{l2} + Q_{r2} \qquad \dot{\theta}_2 = \omega_2$$

$$Q_{r1} = d(\omega_2 - \omega_1) \qquad Q_{r1} = -Q_{l2}$$

where Q_{l1} and Q_{r2} are known time functions and J_1, J_2 and d are parameters.

Example: A Differentiator



$$C\dot{v}_c = i$$

$$v_{out} = K(v_{in} - v_c)$$

$$v_{out} = v_{in} - v_c - Ri$$

If $1/K = 0$, then $v_{out} = -RC\dot{v}_{in}$.

Example continued

Let

$$x = \begin{bmatrix} v_c & v_{out} & i \end{bmatrix}^T$$

and

$$sE - A = \begin{bmatrix} sC & 0 & -1 \\ 1 & -1/K & 0 \\ 1 & 1 & R \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

Example continued

Then

$$(sE - A)\, x(s) = B\, v_{in}(s)$$

so that

$$v_{out}(s) = H(sE - A)^{-1} B\, v_{in}(s) = \frac{RCs}{\frac{RC}{K} s + \frac{K-1}{K}} v_{in}(s)$$

Definition 7

Two pencils $\lambda E_1 - A_1$ and $\lambda E_2 - A_2$ are **strictly equivalent** if there exist square non-singular matrices Q and P such that

$$P(\lambda E_1 - A_1)Q = \lambda E_2 - A_2$$

Notation: Given $A(\lambda) = A_0 + A_1\lambda + \cdots + A_p\lambda^p$ define for a square matrix F of compatible dimension

$$A(F) = A_0 + A_1 F + \cdots + A_p F^p$$

$$\hat{A}(F) = A_0 + F A_1 + \cdots + F^p A_p$$

where $A(F)$ is called the right value of $A(\lambda)$ on substitution of λ with F, and where $\hat{A}(F)$ is called the left value of $A(\lambda)$ on substitution of λ with F.

Theorem 6: Strict Equivalence

The pencils $\lambda I - A$ and $\lambda I - B$ are strictly equivalent if and only if they are equivalent as matrix polynomials.

Proof: Clearly one direction is trivial. Assume that

$$\lambda I - B = P(\lambda)(\lambda I - A)Q(\lambda)$$

where $P(\lambda)$ and $Q(\lambda)$ are invertible. Let $M(\lambda) = P^{-1}(\lambda)$. Then

$$(\lambda I - A)Q(\lambda) = M(\lambda)(\lambda I - B)$$

Proof continued

Since

$$(\lambda I - A)Q(\lambda) = \sum_{i\geq 0} \lambda^{i+1} Q_i - \sum_{i \geq 0} \lambda^i A Q_i$$

it follows that the left value of this polynomial matrix on substitution of λ with A is zero. Hence

$$0 = A\hat{M}(A) - \hat{M}(A)B$$

Let $Q = \hat{M}(A)$. Then $AQ = QB$. If Q is invertible, then with $P = Q^{-1}$ it holds that $PAQ = B$ and

$$P(\lambda I - A)Q = \lambda I - B$$

proving strict equivalence.

Proof continued

Now

$$M(\lambda)P(\lambda) = \sum_j \lambda^j M(\lambda) P_j = I$$

and hence

$$\sum_j A^j \hat{M}(A) P_j = I \Rightarrow \sum_j A^j Q P_j = I$$

From $AQ = QB$ follows $A^j Q = QB^j$ and hence $Q \sum_j B^j P_j = I$. Therefore Q is invertible with inverse $\hat{P}(B)$.

Corollary 2: The pencils $\lambda I - A$ and $\lambda I - B$ are strictly equivalent if and only if A and B are similar.

Definition 8: Block Matrix Notation

For a block diagonal matrix A with blocks A_i, $i = 1, 2, \ldots, s$, introduce the notation

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_s = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{bmatrix}$$

Definition 9: Nilpotent Matrices

A matrix N is **nilpotent** if $N^l = 0$ for some l. The smallest l for which this happens is called the **nilpotency index**.

The matrix

$$N_m = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

of dimension $m \times m$ has nilpotency index m.

Theorem 7: Jordan Normal Form

Any square matrix A is similar to a **Jordan** matrix

$$J = J_{m_1}(\lambda_1) \oplus J_{m_2}(\lambda_2) \oplus \cdots \oplus J_{m_s}(\lambda_s)$$

where λ_i are the eigenvalues of A and where

$$J_m(a) = aI + N_m$$

is a Jordan block of dimension m.

Exercise: Show that

$$\lambda I - J_m(a) \sim \begin{bmatrix} I_{m-1} & 0 \\ 0 & (\lambda - a)^m \end{bmatrix}$$

Also relate the invariant polynomials of $\lambda I - A$ to the Jordan blocks using **elementary divisors**.

Application to Differential Equation

Consider

$$\dot{x}(t) = Ax(t) + f(t)$$

Then there is invertible T such that $T^{-1}AT = J$ where J Jordan matrix. With $g(t) = T^{-1}f(t)$ and $x(t) = Tz(t)$

$$\dot{z}(t) = Jz(t) + g(t)$$

and

$$z(t) = e^{J(t-t_0)}z(t_0) + \int_{t_0}^{t} e^{J(t-\tau)}g(\tau)d\tau$$

where $\exp(Jt) = \exp(J_{m_1}(\lambda_1)t) \oplus \cdots \oplus \exp(J_{m_s}(\lambda_s)t)$, and where it is an **exercise** to compute $\exp(J_{m_i}(\lambda_i)t)$

Lemma 2

Assume that E_1 and E_2 are invertible. Then the two pencils $\lambda E_1 - A_1$ and $\lambda E_2 - A_2$ are strictly equivalent if and only if the two pencils $\lambda I - A_1E_1^{-1}$ and $\lambda I - A_2E_2^{-1}$ are strictly equivalent.

Proof: Assume that there exist invertible P and Q such that

$$P(\lambda E_1 - A_1)Q = \lambda E_2 - A_2$$

Then with $T = P^{-1}$ it holds that

$$T^{-1}A_1E_1^{-1}T = PA_1E_1^{-1}P^{-1} = A_2Q^{-1}E_1^{-1}P^{-1} = A_2E_2^{-1}$$

Assume that there exist invertible T such that $T^{-1}A_1E_1^{-1}T = A_2E_2^{-1}$. Then with $P = T^{-1}$ and $Q = E_1^{-1}TE_2$

$$P(\lambda E_1 - A_1)Q = T^{-1}(\lambda E_1 - A_1)E_1^{-1}TE_2 = \lambda E_2 - T^{-1}A_1E_1^{-1}TE_2 = \lambda E_2 - A_2$$

Theorem 8

Assume that E_1 and E_2 are invertible. Then the two pencils $\lambda E_1 - A_1$ and $\lambda E_2 - A_2$ are strictly equivalent if and only if they are equivalent as matrix polynomials

Proof: Follows from Theorem 6 and Lemma 2.

Corollary 3: When E_1 and E_2 are invertible strict equivalence is equivalent to the pencils having the same invariant polynomials.

Regularity

A pencil $\lambda E - A$ is called **regular** if it is square and $\det(\lambda E - A) \neq 0$ for some λ. Otherwise it is called **singular**.

Lemma 3 If E is invertible then $\lambda E - A$ is regular.

Proof: $\det(\lambda E - A) = \det(\lambda I - AE^{-1})\det E \neq 0$ for any λ which is not an eigenvalue of AE^{-1}.

Example

Consider

$$\lambda E_1 - A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} ; \quad \lambda E_2 - A_2 = \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

They have the same invariant polynomials, 1, 1, λ. Hence they are equivalent as matrix polynomials. They are both regular, but E_1 and E_2 are not invertible, and they are not strictly equivalent, since the ranks of E_1 and E_2 are different.

Conclusion: Equivalence of pencils as matrix polynomials is not necessarily equivalent to strict equivalence of pencils when E_1 and E_1 are not invertible.

Consequence for Linear DAEs

When solving

$$E\dot{x}(t) = Ax(t) + f(t)$$

for noninvertible E it might be necessary to compute derivatives of $f(t)$.

Theorem 9: Weierstrass Normal Form

Any regular pencil $\lambda E - A$ is strictly equivalent to a pencil of the form

$$(I + \lambda N) \oplus (\lambda I - J)$$

where N is a nilpotent matrix and J is a Jordan matrix. Furthermore

$$N = N_{m_1} \oplus N_{m_2} \oplus \cdots \oplus N_{m_r}$$

where the indices m_i will be defined below.

Proof

Since $\lambda E - A$ is regular there is a number c such that $A_1 = cE - A$ is invertible. With $\lambda_1 = \lambda - c$

$$\lambda E - A = \lambda_1 E + A_1$$

Multiplying with A_1^{-1} from the left yields

$$\lambda_1 A_1^{-1} E + I \tag{1}$$

Apply a similarity transformation to $A_1^{-1}E$ so that it is similar to the Jordan matrix $J_0 \oplus J_1$ where J_1 is invertible and J_0 has only zero eigenvalues. Applying the same similarity transformation to (1) yields

$$\lambda_1(J_0 \oplus J_1) + I = (\lambda_1 J_0 + I) \oplus (\lambda_1 J_1 + I) = [(\lambda - c)J_0 + I] \oplus [(\lambda - c)J_1 + I]$$

Proof continued

Now $I - cJ_0$ is invertible and

$$(I - cJ_0)^{-1}[(\lambda - c)J_0 + I] = I + \lambda(I - cJ_0)^{-1}J_0$$

where $(I - cJ_0)^{-1}J_0$ is nilpotent. Therefore $I + \lambda(I - cJ_0)^{-1}J_0$ is strictly equivalent to

$$(I_{m_1} + \lambda N_{m_1}) \oplus (I_{m_2} + \lambda N_{m_2}) \oplus \cdots \oplus (I_{m_r} + \lambda N_{m_r})$$

Since J_1 is invertible, $(\lambda - c)J_1 + I$ is strictly equivalent to $\lambda I - J$ for some Jordan matrix J.

Generalized Eigenvalue Problem

The **generalized eigenvalue** problem associated to the pencil $\lambda E - A$ is

$$(\lambda E - A)x = 0, \; x \neq 0$$

For invertible E this is equivalent to the standard eigenvalue problem for

$$(\lambda I - AE^{-1})Ex = 0, \; z = Ex \neq 0$$

and the eigenvalues are given by either $\det(\lambda I - AE^{-1}) = 0$ or $\det(\lambda E - A) = 0$. The eigenvalues are finite.

For singular E the eigenvalues are given by $\det(\lambda E - A) = 0$. There are also **infinite eigenvalues** which are the inverse of the zero eigenvalues of $(E - \mu A)x = 0, \; x \neq 0$. The indices m_i are the dimensions of the corresponding Jordan blocks.

Application to Linear DAEs

Given

$$E\dot{x}(t) = Ax(t) + f(t)$$

there are non-singular matrices P and Q such that by taking $x(t) = Q(t)$ and $g(t) = Pf(t)$ we get

$$\begin{aligned} \dot{z}_1(t) - Jz_1(t) &= g_1(t) \\ N\dot{z}_2(t) + \quad z_2(t) &= g_2(t) \end{aligned}$$

where N is a nilpotent matrix with nilpotency index $n = \max(m_i)$

n is called the **Differential Index (DI)** of the problem.

KTH

Solution

The solution can be obtained as

$$z_1(t) = e^{J(t-t_0)} z_1(t_0) + \int_{t_0}^{t} e^{J(t-\tau)} g_1(\tau) d\tau \qquad (2)$$

$$z_2(t) = \sum_{i=0}^{n-1} (-N)^i \frac{d^i g_2(t)}{dt^i} \qquad (3)$$

Note:

- Initial values must be consistent with (3)
- The function f must be differentiable

KTH

Differential index of DAEs

Informally, the differential index DI of

$$F(t, \dot{x}, x) = 0 \qquad (4)$$

is the minimum number of times that all or part of (4) must be differentiated with respect to t in order to determine $\dot{x}$ as a continuous function of x.

KTH

Linear Systems II

- System Description by Fractions of Polynomial Matrices
- Poles and Zeros of Multivariable Systems
- Geometrical Theory
- Applications of Geometrical Theory (Disturbance Decoupling)
- Other System Descriptions
- Performance/Robustness Measures