Lecture 6

- Linear Feedback
- Eigenvalue Assignment
- State Observation
- Youla Parameterization

Linear Feedback

Linear System:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases}$$

Linear Feedback:

$$u(t) = K(t)x(t) + N(t)r(t)$$

Closed Loop Linear System:

$$\begin{cases} \dot{x}(t) = [A(t) + B(t)K(t)]x(t) + B(t)N(t)r(t) \\ y(t) = C(t)x(t) \end{cases}$$

Definition

The equation

$$\dot{x}(t) \quad = \quad A(t)x(t)$$

is said to be *uniformly exponentially stable with rate* λ, where $\lambda > 0$, if for $t \geq t_0$

$$\exists \gamma > 0 : \forall t_0, x_0 : \quad |x(t)| \leq \gamma e^{-\lambda(t-t_0)}|x_0|$$

Lemma

The equation $\dot{x}(t) = A(t)x(t)$ is uniformly exponentially stable with rate λ, if and only if the equation

$$\dot{z}(t) \quad = \quad [A(t) - \alpha I]z(t)$$

is uniformly exponentially stable with rate $\lambda + \alpha$.

Proof: The lemma follows from the fact that $x(t)$ solves $\dot{x} = Ax$ if and only if $z(t) = e^{-\alpha t}x(t)$ solves $\dot{z} = [A - \alpha I]z$.

Theorem 1

Suppose that the time invariant system $\{A,B\}$ is controllable. Then, for any $\alpha > \|A\|$, the equation

$$(A+\alpha I)Q + Q(A+\alpha I)^T = BB^T$$

has a solution $Q > 0$ and the feedback $u = -B^T Q^{-1} x$ gives the system

$$\dot{x}(t) = (A - BB^T Q^{-1})x(t)$$

which is uniformly exponentially stable with rate α.

Theorem 2: Eigenvalue Assignment

Suppose $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ with $\{A,B\}$ controllable and rank $B = m$. Then for any degree n polynomial

$$p(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0$$

there is a state feedback $u = Kx$ such that

$$\det(\lambda I - A - BK) = p(\lambda)$$

Proof

Change variables $z = Px$ to controller form:

$$\underbrace{\begin{bmatrix} 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ * & & \dots & * & & \dots & * \\ & & & & 0 & 1 & \\ & & & & & \ddots & \ddots \\ * & \dots & & & & \dots & * \end{bmatrix}}_{A^c} \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 0 \\ 1 & * & \dots & * \\ 0 & 0 & \dots & 0 \\ & & \ddots & \\ 0 & \dots & 0 & 1 \end{bmatrix}}_{B^c}$$

Proof continued

Choose $K = K^c P$, where

$$K^c = \begin{bmatrix} k_{11} & \dots & k_{1n} \\ \vdots & & \vdots \\ k_{m1} & \dots & k_{mn} \end{bmatrix}$$

Start with an input transformation to get $B^c = B_0$, then a state feedback to zero out the $*$'s, and finally a state feedback to get the desired closed-loop characteristic polynomial.

Proof continued 2

Then

$$A^c + B^cK^c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -p_0 & -p_1 & -p_2 & \dots & -p_{n-1} \end{bmatrix}$$

and

$$p(\lambda) = \det(\lambda I - A^c - B^cK^c) = \det(\lambda I - A - BK)$$

Brunovsky Form

Controller form:

$$\dot{x}(t) = (A_o + B_0 U P^{-1})x(t) + B_o R u(t)$$

with the state feedback

$$u(t) = -R^{-1}UP^{-1}x(t) + R^{-1}r(t)$$

yields the closed loop

$$\dot{x}(t) = A_o x(t) + B_o r(t)$$

Brunovsky Form continued

where

$$A_o = \text{block diag}\left\{ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(\rho_i \times \rho_i)}, \quad i = 1, \dots, m \right\}$$

$$B_o = \text{block diag}\left\{ \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T_{(1 \times \rho_i)}, \quad i = 1, \dots, m \right\}$$

and where ρ_i are the controllability indeces.

A_o is nilpotent of index $\max(\rho_i)$

Noninteracting Control

$$\begin{cases} \dot{x}(t) = [A(t) + B(t)K(t)]x(t) + B(t)N(t)r(t) \\ y(t) = C(t)x(t) \end{cases}$$

Choose $K(t)$, $N(t)$ such that

- $r_j(t)$ has no effect on $y_i(t)$ for $i \neq j$, $y \in [t_0, t_f]$.
- The system remains controllable.

Notation

$$C(t) = \begin{bmatrix} C_1(t) \\ \vdots \\ C_p(t) \end{bmatrix}$$

$$L_A[C_i](t) = C_i(t)A(t) + \dot{C}_i(t)$$

$$L_A^{j+1}[C_i](t) = L_A\left(L_A^j[C_i](t)\right) = L_A^j[C_i](t)A(t) + \frac{d}{dt}L_A^j[C_i](t)$$
$$j = 1, 2, 3, 4, \ldots$$
$$L_A^0[C_i](t) = C_i(t)$$

Definition of Relative Degree

The linear system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases}$$

is said to have *constant relative degree* $\kappa_1, \ldots, \kappa_p$ on $[t_0, t_f]$ if

$$\begin{aligned} L_A^j[C_i](t)B(t) &= 0, \quad j = 0, 1, \ldots, \kappa_i - 2 \\ L_A^{\kappa_i - 1}[C_i](t)B(t) &\neq 0 \end{aligned}$$

for $i = 1, \ldots, p$ and $t \in [t_0, t_f]$.

Timeinvariant interpretation!

Theorem 3: Noninteracting Control

Suppose that $p = m$ and $\{A(t), B(t), C(t)\}$ has constant relative degree $\kappa_1, \ldots, \kappa_m$. Let

$$\Omega(t) = \begin{bmatrix} L_A^{\kappa_1 - 1}[C_1](t) \\ \vdots \\ L_A^{\kappa_m - 1}[C_m](t) \end{bmatrix}; \quad \Delta(t) = \begin{bmatrix} L_A^{\kappa_1 - 1}[C_1](t)B(t) \\ \vdots \\ L_A^{\kappa_m - 1}[C_m](t)B(t) \end{bmatrix}$$

If $\Delta(t)$ is invertible at each $t \in [t_0, t_f]$, then $K = -\Delta^{-1}[\Omega A + \dot{\Omega}]$ together with $N = \Delta^{-1}$ achieves noninteracting control. If not, the noninteracting control is not achieved by any $K(t)$, $N(t)$ with $N(t)$ invertible.

Example

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ b(t) & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x \\ L_A^0[C_1]B(t) &= \begin{bmatrix} 0 & 0 \end{bmatrix}; \quad L_A[C_1]B(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \\ L_A^0[C_2]B(t) &= \begin{bmatrix} b(t) & 0 \end{bmatrix} \end{aligned}$$

Example continued

If $b(t) \neq 0$ in $[t_0, t_f]$, then $\kappa_1 = 2$, $\kappa_2 = 1$ and

$$\Delta(t) = \begin{bmatrix} 1 & 1 \\ b(t) & 0 \end{bmatrix}$$

$$u(t) = \begin{bmatrix} 0 & 0 & 1/b(t) & 0 \\ 1 & 1 & -1/b(t) & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1/b(t) \\ 1 & -1/b(t) \end{bmatrix} r(t)$$

State Observation

Goals:

$$\lim_{t\to\infty}[x(t) - \hat{x}(t)] = 0$$
$$x^0 = \hat{x}^0 \quad \Rightarrow \quad x(t) \equiv \hat{x}(t),\ t \geq t_0$$

Necessary Observer Structure:

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + H(t)[y(t) - C(t)\hat{x}(t)]$$

Observer Based Feedback

$$\begin{aligned}
\dot{x} &= Ax + Bu, \quad x(0) = x^0 \\
\dot{\hat{x}} &= A\hat{x} + Bu + H(y - C\hat{x}) \\
u &= K\hat{x} \\
y &= Cx \\
\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & BK \\ HC & A - HC + BK \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \\
\tilde{x} &= x - \hat{x} \\
\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} &= \begin{bmatrix} A + BK & -BK \\ 0 & A - HC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}
\end{aligned}$$

Eigenvalue assignment $\Leftarrow \{A, B\}$ controllable, $\{A, C\}$ observable.

Theorem 4: Reduced Order Observers

Suppose $\{A, C\}$ observable and rank $C = p$. Then given any polynomial $q(\lambda)$ of degree $n - p$, there exists an observer with characteristic polynomial

$$q(\lambda) = \det(\lambda I - E)$$

KTH

Proof Outline

Change variables to

$$\begin{bmatrix} \dot z_a \\ \dot z_b \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} z_a \\ z_b \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} z_a \\ z_b \end{bmatrix}$$

KTH

Proof Outline continued

Use the obsever

$$\begin{aligned} \dot z_c &= (F_{22} - HF_{12}) z_c \\ &\quad + (F_{21} + F_{22}H - HF_{12}H - HF_{11}) y \\ &\quad + (G_2 - HG_1) u \\ \hat z_b &= z_c + Hy \end{aligned}$$

so that $\hat z_b(0) = z_b(0)$ implies $\hat z_b(t) = z_b(t)$.

Then the eigenvalue condition becomes

$$\det(\lambda I - F_{22} + HF_{12}) = q(\lambda)$$

For solvability, it remains to show that $\{F_{22}, F_{12}\}$ is observable. Use PBH.

KTH

Youla Parametrization

Consider the linear system

$$\begin{bmatrix} \dot x \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \quad (1)$$

Suppose that there exist matrices K and L such that $A - KC_2$ and $A - B_2L$ are stable.

KTH

Youla Parametrization continued

Then all stabilizing controllers can be represented on the form

$$\begin{cases} \dot{\hat x} = A\hat x + B_2 u + Ke \\ u = r - L\hat x \\ e = y - C_2 \hat x \end{cases} \quad (2)$$

where $r = Q(s)e$ for some admissible, stable transfer function Q

The closed loop input-output map always takes the form

$$T_{zw}(s) + T_{zr}(s)Q(s)T_{ew}(s),$$

where T_{zw}, T_{zr} and T_{ew} are stable proper transfer matrices defined by the plant and K and L.

Proof in easy direction

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \\ z \\ e \end{bmatrix} = \begin{bmatrix} A-B_2L & B_2L & B_1 & B_2 \\ 0 & A-KC_2 & B_1-KD_{21} & 0 \\ C_1-D_{12}L & D_{12}L & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \\ w \\ r \end{bmatrix}$$

Note that $w \equiv 0$ and $\tilde{x}(0) = 0$ gives $\tilde{x} \equiv 0$ and $e \equiv 0$, so the transfer function from r to e is zero. Therefore

$$\begin{bmatrix} z \\ e \end{bmatrix} = \begin{bmatrix} T_{zw} & T_{zr} \\ T_{ew} & 0 \end{bmatrix} \begin{bmatrix} w \\ r \end{bmatrix}.$$

Application

Design or optimization of $Q(s)$ is often more tractable than the corresponding problem for a general $K(s)$.

For example,

$$\min_Q \|T_{zw} + T_{zr} Q T_{ew}\|$$

is a convex problem.

Next Week

- Polynomial matrices and Smith normal form
- Differential Algebraic Equations and Pencils
- Weierstrass normal form and Kronecker normal form
- Introduction to Polynomial Fraction Description