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Lecture 4

- Controllability
- Observability
- Controller and Observer Forms
- Balanced Realizations

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Controllability

The equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x^0$$

is called *controllable on* (t_0, t_f), if for any x^0, there exists $u(t)$ such that $x(t_f) = 0$. The matrix function

$$W(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, t)B(t)B(t)^T\Phi(t_0, t)^T dt$$

is called *controllability Gramian.*

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Operator Interpretation

Define $M : \mathbf{L}_2^m[t_0, t_f] \to \mathbf{R}^n$ by

$$Mu = \int_{t_0}^{t_f} \Phi(t_0, \tau)B(\tau)u(\tau)d\tau$$

Then

$$\begin{aligned} x(t_f) &= \Phi(t_f, t_0)[x(t_0) + Mu] \\ (M^*x)(t) &= B(t)^T\Phi(t_0, t)^T x \\ MM^* &= \int_{t_0}^{t_f} \Phi(t_0, \tau)B(\tau)B(\tau)^T\Phi(t_0, \tau)^T d\tau \\ &= W(t_0, t_f) \end{aligned}$$

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Degree of Controllability

The minimal input, in terms of $|u|$, to go from $x(t_0) = x_0$ to $x(t_f) = 0$ can be used to evaluate degree of controllability.

From Lecture 2: Minimize $|u|$ under the constraint $x_0 + Mu = 0$.

$$\hat{u} = -M^*(MM^*)^{-1}x_0 \quad \text{(if } MM^* \text{ invertible)}$$

$$\begin{aligned} |\hat{u}|^2 &= x_0^T(MM^*)^{-1}x_0 \\ &= x_0^T W(t_0, t_f)^{-1}x_0 \end{aligned}$$

Theorem 1: Controllability Criterion

The system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ is controllable on (t_0, t_f) if and only if $W(t_0, t_f) > 0$. The minimal cost $\int_{t_0}^{t_f} |u|^2 dt$ to reach 0 from x_0 is $x_0^T W(t_0, t_f)^{-1} x_0$.

Proof of Theorem 1

Controllability on (t_0, t_f)

$$
\begin{aligned}
&\Leftrightarrow \quad \forall x_f : \exists u : x(t_f) = 0 \\
&\Leftrightarrow \quad \forall x_f : \exists u : x_0 + Mu = 0 \\
&\Leftrightarrow \quad \mathcal{R}(M) = \mathbf{R}^n \\
&\Leftrightarrow \quad \mathcal{N}(M^*) = \{0\} \\
&\Leftrightarrow \quad \mathcal{N}(MM^*) = \{0\} \\
&\Leftrightarrow \quad \mathcal{N}[W(t_0, t_f)] = \{0\} \\
&\Leftrightarrow \quad W(t_0, t_f) > 0
\end{aligned}
$$

Theorem 2: Time-Invariant Controllability

The following four conditions are equivalent:

(i) The system $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable.

(ii) $\mathrm{rank}[B\, AB\, A^2B\, \ldots\, A^{n-1}B] = n$.

(iii) $\lambda \in \mathbf{C}$, $p^T A = \lambda p^T$, $p^T B = 0 \quad \Rightarrow p = 0$ (PBH-test)

(iv) rank $[\lambda I - A \quad B] = n \quad \forall \lambda \in \mathbf{C}$. (PBH-test)

Popov-Belevitch-Hautus (PBH), see p221.

Notice the Rugh Example 9.6 on p147.

Theorem 3: Uncontrollable State Equation

Suppose that $0 < q < n$ and

$$\mathrm{rank}\begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} = q < n$$

Then there exists an invertible $P \in \mathbf{R}^{n \times n}$ such that

$$P^{-1}AP = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{22} \end{bmatrix} \qquad P^{-1}B = \begin{bmatrix} \widehat{B}_{11} \\ 0 \end{bmatrix}$$

where $\widehat{A}_{11}$ is $q \times q$, $\widehat{B}_{11}$ is $q \times m$, and

$$\mathrm{rank}[\widehat{B}_{11} \quad \widehat{A}_{11}\widehat{B}_{11} \ldots \widehat{A}_{11}^{q-1}B_{11}] = q$$

Proof of Theorem 3

Let $p_1, \dots, p_q$ be linearly independent columns from

$$[B \quad AB \dots A^{n-1}B]$$

Let $p_{q+1} \dots p_n$ be additional columns that make

$$P = [p_1 \dots p_q \, p_{q+1} \dots p_n]$$

invertible.

Proof continued

With $[\widehat{A}_1 \ \widehat{A}_2] = P^{-1}AP, \widehat{B} = P^{-1}B$

$$\mathcal{R}(P\widehat{B}) = \mathcal{R}(B) \subset \mathcal{R}([p_1 \dots p_q])$$

$$\Rightarrow \quad \widehat{B} = \begin{bmatrix} \widehat{B}_1 \\ 0 \end{bmatrix}$$

$$\mathcal{R}(P\widehat{A}_1) = \mathcal{R}(A[p_1 \dots p_q]) \subset \mathcal{R}([p_1 \dots p_q])$$

$$\Rightarrow \quad \widehat{A}_1 = \begin{bmatrix} \widehat{A}_{11} \\ 0 \end{bmatrix}$$

Proof of Theorem 2

(i) $\Rightarrow$ **(ii)** If (ii) fails, then after a coordinate change

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = Px$$

as in Theorem 3, $\hat{x}_2$ is unaffected by the input, so (i) fails.

(ii) $\Rightarrow$ **(i)** If $p^T W(t_0, t_f) p = 0$ for some $p \neq 0$, then

$$p^T e^{A(t_0 - t)} B = 0 \quad \forall t \in [t_0, t_f]$$

Differentiation with respect to t at $t = t_0$, gives

$$p^T [B \quad AB \dots A^{n-1}B] = 0,$$

so (ii) fails.

Proof continued

(ii) $\Rightarrow$ **(iii)** If $p^T A = \lambda p^T$ and $p^T B = 0$ then $p^T [B \quad AB \dots A^{n-1}B] = 0$, so **(ii)** fails.

(iii) $\Rightarrow$ **(ii)** If $\operatorname{rank}[B \ \dots \ A^{n-1}B] = q < n$ then let P be defined as in Theorem 3 and let ${p_2}^T \hat{A}_{22} = \lambda {p_2}^T$ and $p^T = [0 \quad {p_2}^T] P^{-1}$.

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Proof continued

Then

$$p^T B = [0 \quad {p_2}^T] \begin{bmatrix} \hat{B}_{11} \\ 0 \end{bmatrix} = 0$$

$$p^T A = [0 \quad {p_2}^T] \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} P^{-1} = \lambda[0 \quad {p_2}^T]P^{-1} = \lambda p^T$$

so **(iii)** fails.

(iv) $\Leftrightarrow \{p^T[\lambda - A \quad B] = 0 \ \Rightarrow \ p = 0\} \Leftrightarrow$ **(iii)**

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Reachability

The equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = 0$$

is called *reachable on* (t_0, t_f), if for any x_f, there exists $u(t)$ such that $x(t_f) = x_f$. The matrix function

$$W_r(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_f, t)B(t)B(t)^T \Phi(t_f, t)^T dt = \Phi(t_f, t_0)W(t_0, t_f)\Phi(t_f, t_0)^T$$

is called *reachability Gramian*. If $A(t)$ is continuous, then controllability and reachability are equivalent.

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Observability

The equation

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x^0$$
$$y(t) = C(t)x(t)$$

is called *observable on* $[t_0, t_f]$ if any initial state x^0 is uniquely determined by the output $y(t)$ for $t \in [t_0, t_f]$.

It is called *reconstructable on* $[t_0, t_f]$ if the state $x(t_f)$ is uniquely determined by the output $y(t)$ for $t \in [t_0, t_f]$.

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Observability Gramian

The matrix function

$$M(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t, t_0)^T C(t)^T C(t)\Phi(t, t_0)dt$$

is called the *observability Gramian* of the system

$$\dot{x}(t) = A(t)x(t)$$
$$y(t) = C(t)x(t)$$

Operator interpretation:

$$M(t_0, t_f) = L^* L$$

where $L : \mathbf{R}^n \to L_2^m(t_0, t_f)$ with

$$(Lx^0)(t) = C(t)\Phi(t, t_0)x^0, \quad x^0 \in \mathbf{R}^n$$

Theorem 4: Observability Criterion

The following two conditions are equivalent

(i) The system defined by $\{A(t), C(t)\}$ is observable on $[t_0, t_f]$.

(ii) $M(t_0, t_f) > 0$

Degree of Observability Consider $y = Lx^0 + e$, where e is white noise with unit variance.

$|y - Lx^0|$ is minimized for $\hat{x}^0 = (L^*L)^{-1}L^*y$ and the variance of the estimate is

$(L^*L)^{-1} = M(t_0, t_f)^{-1}$.

Theorem 5: Time-Invariant Observability

The following four conditions are equivalent:

(i) The system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ is observable.

(ii) $\operatorname{rank} \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$

(iii) $\nexists p \in \mathbf{C}^n, \lambda \in \mathbf{C} : \; Ap = \lambda p, \; Cp = 0$ (PBH-test)

(iv) $\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathbf{C}.$ (PBH-test)

Theorem 6: Unobservable State Equation

Suppose that $0 < l < n$ and $\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = l < n$. Then there exists an invertible $Q \in \mathbf{R}^{n \times n}$ such that

$$Q^{-1}AQ = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \qquad CQ = \begin{bmatrix} \hat{C}_{11} & 0 \end{bmatrix}$$

where $\hat{A}_{11}$ is $l \times l$, $\hat{C}_{11}$ is $p \times l$, and the system $\dot{x}(t) = \hat{A}_{11}x(t)$, $y(t) = \hat{C}_{11}x(t)$ is observable.

Definition: Controllability Index

Let $B = [B_1 \; \ldots \; B_m]$. For $j = 1, \ldots, m$, the *controllability index* ρ_j is the smallest integer such that $A^{\rho_j}B_j$ is linearly dependent on the column vectors occuring to the left of it in the controllability matrix

$$\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$$

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Notation for Controller Form

Given a pair $\{A, B\}$, with controllability indices $\rho_1, \dots \rho_m$, define

$$M = \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix} := \begin{bmatrix} B_1 & AB_1 \dots A^{\rho_1 - 1}B_1 & \dots & B_m \dots A^{\rho_m - 1}B_m \end{bmatrix}^{-1}$$

$$P^c = \begin{bmatrix} P_1 \\ \vdots \\ P_m \end{bmatrix}, \quad P_i = \begin{bmatrix} M_{\rho_1 + \dots + \rho_i} \\ M_{\rho_1 + \dots + \rho_i} A \\ \vdots \\ M_{\rho_1 + \dots + \rho_i} A^{\rho_i - 1} \end{bmatrix}$$

Notice that it is rather easy to write Matlab code for this.

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Theorem 7, Controller form

Suppose $\dot{x} = Ax + Bu$ is controllable, with controllability indices $\rho_1, \dots \rho_m$. Then the variable transformation $z = P^c x$ gives $\dot{z} = A^c z + B^c u$ with

$A^c =$ $\qquad$ $B^c =$

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Comments

Reverse statevariable ordering in each block to get the SISO controller form of AK.
Reveals Structure, Pole Placement,
Minimal Realizations

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Definition: Observability Index

Let $C^T = [C_1{}^T \ \dots \ C_p{}^T]^T$. For $j = 1, \dots, p$, the *observability index* η_j is the smallest integer such that $C_j A^{\eta_j}$ is linearly dependent on the row vectors occuring above it in the observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Theorem 8: Observer form

Suppose $\dot{x} = Ax, y = Cx$ is observable, with observability indices $\eta_1, \ldots \eta_p$. Then the variable transformation $z = P^o x$ gives $\dot{z} = A^o z + B^o u$ with

$A^o =$

$C^o =$

Theorem 9: Time-Invariant Gramian

Let A be exponentially stable. Then, the reachability Gramian $W_c(-\infty, 0)$ equals the unique solution P to the matrix equation

$$PA^T + AP = -BB^T$$

Similarly, the observability Gramian $M(0, \infty)$ equals the solution Q of

$$QA + A^TQ = -C^TC$$

Proof of Theorem 9

Let $P = W_r(-\infty, 0) = \int_0^\infty e^{A\sigma}BB^Te^{A^T\sigma}d\sigma$. Then

$$\begin{aligned} PA^T + AP &= \int_0^\infty \frac{\partial}{\partial\sigma}\left(e^{A\sigma}BB^Te^{A^T\sigma}\right)d\sigma \\ &= \left[e^{A\sigma}BB^Te^{A^T\sigma}\right]_0^\infty = -BB^T \end{aligned}$$

The linear operator

$$L(P) = AP + PA^T$$

has $\mathcal{R}(L) = \mathbf{R}^{n\times n}$ so $\mathcal{N}(L) = \{0\}$ and the solution P is unique. (Lyapunov 1893)

The equation for the observability Gramian is obtained by replacing A, B with A^T, C^T.

Balanced Realization

For the stable system (A, B, C), with Gramians P and Q, the variable transformation $\hat{x} = Tx$ gives

$$\hat{P} = TPT^*, \qquad \hat{Q} = T^{-*}QT^{-1}$$

The choice of quadratic R, T, unitary U and diagonal Σ such that

$$\begin{aligned} Q &= R^*R \quad \text{(Choleski Factorisation)} \\ RPR^* &= U\Sigma^2U^* \quad \text{(Singular Value Decomposition)} \\ T &= \Sigma^{-1/2}U^*R \end{aligned}$$

gives

$$\hat{P} = \hat{Q} = \Sigma$$

$(\hat{A}, \hat{B}, \hat{C})$ is called a *balanced realization* of the system (A, B, C).

Truncated Balanced Realization

Let the states be sorted such that Σ is decreasing. The diagonal elements of Σ measures "how controllable and observable" the corresponding states are. With

$$\widehat{A} = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{bmatrix}$$

$$\widehat{C} = \begin{bmatrix} \widehat{C}_1 & \widehat{C}_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$(\widehat{A}_{11}, \widehat{B}_1, \widehat{C}_1)$ is called a *truncated balanced realization* of (A, B, C).

See also R. Johansson: System Modeling & Identification, p236

Example

$$C(sI - A)^{-1}B = \frac{1 - s}{s^6 + 3s^5 + 5s^4 + 7s^3 + 5s^2 + 3s + 1}$$

$$\Sigma = \mathrm{diag}\{1.98, 1.92, 0.75, 0.33, 0.15, 0.0045\}$$

$$\widehat{C}(sI - \widehat{A})^{-1}\widehat{B} = \frac{0.20s^2 - 0.44s + 0.23}{s^3 + 0.44s^2 + 0.66s + 0.17}$$

(Done with `balreal` in MATLAB!)

Next week

- Realization from Weighting Pattern
- Realization from Impulse Response
- Realization from Markov Parameters
- Minimal Realizations