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Lecture 3

- More on Adjoints
- Periodic Systems
- Internal Stability
- Discrete Time Systems

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Properties of the Adjoint

Let M be a bounded linear operator between two real Hilbert spaces. Then

$$
\begin{aligned}
M^{**} &= M && (1)\\
[\mathcal{R}(M)]^{\perp} &= \mathcal{N}(M^*) && (2)\\
\overline{\mathcal{R}(M)} &= [\mathcal{N}(M^*)]^{\perp} && (3)\\
[\mathcal{R}(M^*)]^{\perp} &= \mathcal{N}(M) && (4)\\
\overline{\mathcal{R}(M^*)} &= [\mathcal{N}(M)]^{\perp} && (5)\\
\mathcal{N}(M^*) &= \mathcal{N}(MM^*) && (6)\\
\mathcal{N}(M) &= \mathcal{N}(M^*M) && (7)
\end{aligned}
$$

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Proof of (7)

$$
\begin{aligned}
Mx = 0 &\Rightarrow M^*Mx = 0\\
&\Rightarrow 0 =< x, M^*Mx >=< Mx, Mx >\\
&\Rightarrow Mx = 0
\end{aligned}
$$

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Example: Shift Operator on l_2

$$
\begin{aligned}
l_2 &= \{x = (x_1, x_2, x_3, \dots) : \textstyle\sum_{i=1}^{\infty} {x_i}^2 < \infty\}\\
x &= (x_1, x_2, x_3, \dots)\\
y &= (y_1, y_2, y_3, \dots)\\
Sx &= (0, x_1, x_2, \dots)\\
S^*y &= (y_2, y_3, \dots)\\
< y, Sx > &= \textstyle\sum_{i=1}^{\infty} y_{i+1}x_i =< S^*y, x >\\
\mathcal{R}(S) &= \{(0, *, *, *, \dots)\}\\
\mathcal{N}(S^*) &= \{(*, 0, 0, 0, \dots)\}
\end{aligned}
$$

Example: Observability Gramian

For $x^0 \in \mathbf{R}^n$, $y \in \mathbf{L}_2^m[t_0, t_1]$, introduce

$$(Mx^0)(t) = C(t)\Phi(t,t_0)x^0, \quad t \in [t_0, t_1]$$

$$M^* y = \int_{t_0}^{t_1} \Phi(t,t_0)^T C(t)^T y(t) dt$$

Then the "unobservable" initial states can be computed as

$$\mathcal{N}(M) = \mathcal{N}(M^*M) = \mathcal{N}\left(\int_{t_0}^{t_1} \Phi(t,t_0)^T C(t)^T C(t)\Phi(t,t_0) dt\right)$$

$$\int_{t_0}^{t_1} \Phi(t,t_0)^T C(t)^T C(t)\Phi(t,t_0) dt$$

will later be called the observability Gramian of the system.

Floquet Decomposition

Let $A(t)$ be continuous and T-periodic. Then for

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x^0$$

the transition matrix can be written

$$\Phi(t,\tau) = P(t)e^{R(t-\tau)}P(\tau)^{-1}$$

where $R \in \mathbf{C}^{n\times n}$ is constant and $P(t)$ is continuous and T-periodic.

With the variable transformation $x(t) = P(t)z(t)$, this gives $\dot{z} = Rz$.

Proof

Define R and $P(t)$ by

$$\begin{aligned} e^{RT} &= \Phi(T,0) \\ P(t) &= \Phi(t,0)e^{-Rt} \end{aligned}$$

Then

$$\begin{aligned} \Phi(t,\tau) &= \Phi(t,0)\Phi(\tau,0)^{-1} = P(t)e^{R(t-\tau)}P(\tau)^{-1} \\ P(t+T) &= \Phi(t+T,0)e^{-R(t+T)} \\ &= \Phi(t+T,T)\Phi(T,0)e^{-RT}e^{-Rt} \\ &= \Phi(t+T,T)e^{-Rt} \\ &= \Phi(t,0)e^{-Rt} \\ &= P(t) \end{aligned}$$

Example: Sinusodal Input

Consider the equation

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \sin t \end{bmatrix} \qquad x(0) = x^0$$

Laplace transform:

$$\begin{aligned} \mathbf{x}_2(s) &= C(sI - A)^{-1}(Bu(s) + x^0) = \frac{s}{(1+s^2)^2} + \frac{1}{1+s^2}\begin{bmatrix} 1 & s \end{bmatrix} x^0 \\ x_2(t) &= \frac{t}{2}\sin t + \begin{bmatrix} \sin t & \cos t \end{bmatrix} x^0 \end{aligned}$$

For what systems does periodic input give periodic solution?

Condition for Periodic Solutions

Let $A(t)$ be continuous and T-periodic and

$$\dot{x}(t) = A(t)x(t) + f(t)$$

The following statements are then equivalent:

(i) No nontrivial T-periodic solution exists for $f \equiv 0$.

(ii) A unique T-periodic solution exists for every T-periodic f.

Proof

$$\begin{aligned}
\text{(i)} \quad &\Leftrightarrow \quad \Phi(t_0+T,t_0)x^0 = x^0 \quad \Rightarrow x^0 = 0 \\
&\Leftrightarrow \quad \det\left(\Phi(t_0+T,t_0) - I\right) \neq 0 \\
&\Leftrightarrow \quad \forall f:\ \exists x^0: \\
&\qquad x^0 = \Phi(t_0+T,t_0)x^0 + \int_{t_0}^{t_0+T} \Phi(t_0+T,\tau)f(\tau)d\tau \\
&\Leftrightarrow \quad \text{(ii)}
\end{aligned}$$

Corollary: LTI Systems

For $A \in \mathbf{R}^{n\times n}$ and

$$\dot{x}(t) = Ax(t) + f(t), \quad x(t_0) = x^0$$

the following statements are equivalent:

(i) No eigenvalue of A has zero real part.

(ii) A unique T-periodic solution exists for every T-periodic f.

Definition of Uniform Stability

The system $\dot{x}(t) = A(t)x(t)$ is called

uniformly stable if $\exists \gamma > 0$ such that

$$|x(t)| \quad < \quad \gamma|x(t_0)| \quad \forall t \geq t_0 \geq 0$$

uniformly asymptotically stable if it is uniformly stable and $\forall \delta > 0:\ \exists T > 0:$

$$|x(t)| \quad < \quad \delta|x(t_0)| \quad \forall t \geq t_0 + T$$

uniformly exponentially stable if $\exists \gamma, \lambda$ such that for $t \geq t_0 \geq 0$ one has

$$|x(t)| \quad < \quad \gamma|x(t_0)|e^{-\lambda(t-t_0)}$$

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Transition Matrix Conditions

The system $\dot{x}(t) = A(t)x(t)$ is

uniformly stable if $\exists \gamma > 0$ such that

$$\|\Phi(t, t_0)\| \quad < \quad \gamma \quad \forall t \geq t_0 \geq 0$$

uniformly asymptotically stable if it is uniformly stable and $\forall \delta > 0 : \; \exists T > 0 :$

$$\|\Phi(t, t_0)\| \quad < \quad \delta \quad \forall t \geq t_0 + T$$

uniformly exponentially stable if $\exists \gamma, \lambda$ such that for $t \geq t_0 \geq 0$ one has

$$\|\Phi(t, t_0)\| \quad < \quad \gamma e^{-\lambda(t-t_0)}$$

Proof: Use $x(t) = \Phi(t, t_0)x(t_0)$ and the definition of matrix norm.

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Criterion for Exponential Stability

For the equation $\dot{x}(t) = A(t)x(t)$ with $\|A(t)\|$ bounded, the following three conditions are equivalent:

(i) The equation is uniformly exponentially stable.

(ii) The equation is uniformly asymptotically stable.

(iii) There exists a $\beta > 0$ such that

$$\int_\tau^t \|\Phi(t, \sigma)\| d\sigma \leq \beta \quad \forall \tau \leq t$$

Proof: (i) $\Rightarrow$ (iii) is obvious.

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Proof of (iii) $\Rightarrow$ (ii)

Let $\alpha = \sup_t \|A(t)\|$. Then

$$\begin{aligned}\|\Phi(t, \tau)\| &= \left\| I - \int_\tau^t \frac{\partial}{\partial \sigma} \Phi(t, \sigma) d\sigma \right\| = \left\| I + \int_\tau^t \Phi(t, \sigma) A(\sigma) d\sigma \right\| \\ &\leq 1 + \alpha \int_\tau^t \|\Phi(t, \sigma)\| d\sigma \leq 1 + \alpha\beta\end{aligned}$$

$$\begin{aligned}\|\Phi(t, \tau)\| &= \frac{1}{t - \tau} \int_\tau^t \|\Phi(t, \tau)\| d\sigma \\ &\leq \frac{1}{t - \tau} \int_\tau^t \|\Phi(t, \sigma)\| \cdot \|\Phi(\sigma, \tau)\| d\sigma \leq \frac{\beta}{t - \tau}(1 + \alpha\beta)\end{aligned}$$

which proves asymptotic stability.

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Proof of (ii)$\Rightarrow$ (i)

Assume asymptotic stability. To prove exponential stability, select $\gamma, T > 0$ such that

$$\|\Phi(t, t_0)\| \quad \leq \quad \gamma \quad \forall t \geq t_0; \quad \|\Phi(t_0 + T, t_0)\| \leq \frac{1}{2} \quad \forall t \geq t_0 + T$$

Then

$$\begin{aligned}&\|\Phi(t_0 + kT, t_0)\| \leq \|\Phi(t_0 + kT, t_0 + (k-1)T)\| \cdots \|\Phi(t_0 + T, t_0)\| \\ &\quad \leq \frac{1}{2^k} \quad k = 1, 2, \ldots \\ &\|\Phi(t, t_0)\| \leq \|\Phi(t, t_0 + kT)\| \cdot \|\Phi(t_0 + kT, t_0)\| \leq \frac{\gamma}{2^k} \quad t \geq t_0 + kT\end{aligned}$$

This proves exponential stability with $\lambda = \ln 2/2T$.

Example: Stability by Coordinate Change

Note that the scalar system

$$\dot{x} = x$$

is not stable, but the change of coordinates $z(t) = e^{-2t}x(t)$ gives the stable equation

$$\dot{z} = -z$$

Lyapunov Transformation

An $n \times n$ continuously differentiable matrix function is called a *Lyapunov transformation* if there exist $\rho > 0$ such that

$$\|P(t)\| \le \rho, \quad \|P(t)^{-1}\| \le \rho \quad \forall t$$

Stability Preserved

Both uniform stability and uniform exponential stability are preserved under a coordinate transformation $x(t) = P(t)z(t)$ defined by a Lyapunov transformation.

Proof. This follows immediately from the relations

$$\begin{aligned} \|\Phi_x(t,t_0)\| &= \|P(t)\Phi_z(t,t_0)P(t_0)^{-1}\| \\ &\le \rho^2\|\Phi_z(t,t_0)\| \end{aligned}$$

$$\begin{aligned} \|\Phi_z(t,t_0)\| &= \|P(t)^{-1}\Phi_x(t,t_0)P(t_0)\| \\ &\le \rho^2\|\Phi_x(t,t_0)\| \end{aligned}$$

Discrete Time Systems

Given a matrix sequence $A(0), A(1), \ldots$ the equation

$$x(k+1) = A(k)x(k), \quad x(k_0) = x^0$$

has the unique solution

$$x(k) = \Phi(k,k_0)x^0$$

defined by the *transition matrix*

$$\Phi(k,k_0) = \begin{cases} A(k-1)\cdots A(k_0), & k > k_0 \\ I, & k = k_0 \end{cases}$$

Proof by inspection.

Input-driven System

The equation

$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$x(k_0) = x^0$$

has the unique solution

$$x(k) = \Phi(k,k_0)x^0 + \sum_{k_0}^{k-1} \Phi(k,\sigma)B(\sigma)u(\sigma)$$

Proof by inspection.

Recursive definintion of $\Phi(k, k_0)$

Define $X(k)$ recursively as

$$X(k+1) = A(k)X(k), \quad k \geq k_0$$
$$X(k_0) = I$$

Then $\Phi(k,k_0) = X(k)$.

Properties of $\Phi(k, k_0)$

$$\Phi(k+1,j) = A(k)\Phi(k,j), \quad k \geq j$$
$$\Phi(k,j-1) = \Phi(k,j)A(j-1), \quad k \geq j$$
$$\Phi(k,i) = \Phi(k,j)\Phi(j,i), \quad i \leq j \leq k$$

Inversion

If the $n \times n$ matrix $A(k)$ is invertible for each k, then $\Phi(k,j)$ is invertible for each $k \geq j$ and $\Phi(j,k)$ can be defined as

$$\Phi(j,k) = \Phi(k,j)^{-1}$$

Change of Variables

The equation

$$x(k+1) = A(k)x(k), \quad x(k_0) = x^0$$

with new variables

$$x(k) = P(k)z(k)$$

writes

$$z(k+1) = \left[P(k+1)^{-1}A(k)P(k)\right] z(k)$$

and has transition matrix

$$\Phi_z(k,j) = P(k)^{-1}\Phi_x(k,j)P(j)$$

Floquet Decomposition

Let $A(k)$ be K-periodic. Then for

$$x(k+1) = A(k)x(k), \quad x(k_0) = x^0$$

the transition matrix can be written

$$\Phi(k,j) = P(k)R^{(k-j)}P(j)^{-1}$$

where $R \in \mathbf{C}^{n\times n}$ and $P(k)$ is K-periodic.

With $x(k) = P(k)z(k)$, this gives

$$z(k+1) = Rz(k)$$

2-periodic Example

$$A(k) = \begin{bmatrix} (-1)^k & 0 \\ 0 & 1 \end{bmatrix}$$

$$R^2 = \Phi(2,0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$$

Condition for Periodic Solutions

Let $A(k)$ be K-periodic and

$$x(k+1) = A(k)x(k) + f(k)$$

The following statements are then equivalent:

(i) No nontrivial K-periodic solution exists for $f \equiv 0$.

(ii) A unique K-periodic solution exists for every K-periodic f.

Proof. Analogous to continuous time.

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Next Lecture

- Controllability and Observability
- Controller and Observer forms
- Gramians
- Balanced Realizations