Linear Systems I, 1999

- Introduction
- Multivariable Time-varying Systems
- Transition Matrices
- Controllability and Observability
- Realization Theory
- Linear Feedback
- Multivariable input/output descriptions

Rugh, Linear System Theory,
chapter 1-15, 20-29

Lecture 1

- Examples
- Linearization
- Examples
- Mathematical Concepts
- Block diagrams
- Looking Forward

Linear Time-invariant System

State Representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad x(0) = 0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Convolution Representation

$$\begin{aligned} y(t) &= \int_0^t G(t-\tau)u(\tau)d\tau \\ G(t) &= Ce^{At}B + \delta(t)D \quad \text{(impulse response)} \end{aligned}$$

Transfer Function Representation

$$\begin{aligned} \mathbf{y}(s) &= \mathbf{G}(s)\mathbf{u}(s) \\ \mathbf{G}(s) &:= \int_{0-}^{\infty} e^{-st}G(t)dt = C(sI-A)^{-1}B + D \end{aligned}$$

Time-varying Linear System

State Representation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \qquad x(0) = 0 \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

Integral Representation

$$y(t) = \int_0^t G(t,\tau)u(\tau)d\tau + D(t)u(t)$$

Operator Representation

$$y = Lu$$

Example: Two Tank System

Flow: $q(t)$

Volumes: V_1, V_2

Concentrations: $u(t)$, $x_1(t)$, $x_2(t)$

Dynamics

$$\begin{cases} \frac{d}{dt}(V_1 x_1) & = & qu - qx_1 \\ \frac{d}{dt}(V_2 x_2) & = & qx_1 - qx_2 \end{cases}$$

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{V_1} & 0 \\ \frac{1}{V_2} & -\frac{1}{V_2} \end{bmatrix} q(t)x(t) + \begin{bmatrix} \frac{1}{V_1} \\ 0 \end{bmatrix} q(t)u(t)$$

Example: Electric Circuit

Capacitor Dynamics:

$$i(t) = \frac{d}{dt}(c(t)u_c(t))$$

Inductor Dynamics:

$$u_l(t) = \frac{d}{dt}(l(t)i(t))$$

State Representation

With $x = [u_c \; i]^T$

$$\dot{x}(t) = \begin{bmatrix} -\dot{c}/c & 1/c \\ -1/l & -\left(r + \dot{l}\right)/l \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/l \end{bmatrix} u(t)$$

Discrete Time LTI System

State Representation

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \qquad x(0) = 0 \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Convolution Representation

$$\begin{aligned} y(k) &= \sum_{0}^{k} G(k-l)u(l) \\ G(k) &= \begin{cases} D & k = 0 \\ CA^{k-1}B & k \geq 1 \end{cases} \qquad \text{(impulse response)} \end{aligned}$$

Transfer Function Representation

$$\begin{aligned} \mathbf{y}(z) &= \mathbf{G}(z)\mathbf{u}(z) \\ \mathbf{G}(z) &:= \sum_{k=0}^{\infty} G(k)z^{-k} = C(zI - A)^{-1}B + D \end{aligned}$$

Example: Shift Register

State Representation

$$\begin{aligned} x &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \\ x(k+1) &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x(k) \end{aligned}$$

Linearization

Consider (assuming differentiability of f)

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0$$

with solution $\tilde{x}$ for $u = \tilde{u}$ and $x_0 = \tilde{x}_0$. Let $x_\delta = x - \tilde{x}$. Then

$$\begin{aligned} & f(\tilde{x} + x_\delta, \tilde{u} + u_\delta, t) - f(\tilde{x}, \tilde{u}, t) \\ &= \frac{\partial f}{\partial x}(\tilde{x}, \tilde{u}, t)x_\delta + \frac{\partial f}{\partial u}(\tilde{x}, \tilde{u}, t)u_\delta + o(|x_\delta|, |u_\delta|) \end{aligned}$$

Hence, with

$$A(t) = \frac{\partial f}{\partial x}(\tilde{x}, \tilde{u}, t), \quad B(t) = \frac{\partial f}{\partial u}(\tilde{x}, \tilde{u}, t)$$

the linearization is

$$\dot{x}_\delta(t) = A(t)x_\delta(t) + B(t)u_\delta(t), \quad x_\delta(0) = x_0 - \tilde{x}_0$$

Example: Communications Satellite

Spherical coordinates: $x = [r\ \dot{r}\ \theta\ \dot{\theta}\ \phi\ \dot{\phi}]^T$

Input: $u = [u_r\ u_\theta\ u_\phi]^T$

Output: $y = [r\ \theta\ \phi]^T$

Dynamics

$$\dot{x}(t) = f(x(t), u(t), t)$$

$$= \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 \cos^2\phi + r\phi^2 - k/r^2 + u_r/m \\ \dot{\theta} \\ -2\dot{r}\dot{\theta}/r + 2\dot{\theta}\dot{\phi}\sin\phi/\cos\phi + u_\theta \cos\phi/mr \\ \dot{\phi} \\ -\dot{\theta}^2\cos\phi\sin\phi - 2\dot{r}\dot{\phi}/r + u_\phi/mr \end{bmatrix}$$

Linearized Communications Satellite

Circular equatorial orbit:

$$\begin{aligned} \tilde{x} &= \begin{bmatrix} \tilde{r} & 0 & \tilde{\omega}t & \tilde{\omega} & 0 & 0 \end{bmatrix}^T \\ \tilde{u} &\equiv 0 \end{aligned}$$

Linearization: $\dot{x} = Ax + Bx$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\tilde{\omega}^2 & 0 & 0 & 2\tilde{\omega}\tilde{r} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2\tilde{\omega}/\tilde{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\tilde{\omega}^2 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1/m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/m\tilde{r} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/m\tilde{r} \end{bmatrix}$$

KTH

Vector Functions

The notation $x \in \mathbf{L}_2^n[0,\infty)$ means that

$$\begin{cases} x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbf{R}^n, \quad t \in [0,\infty) \\ \int_0^\infty x_i(t)^2 dt < \infty, \quad \forall i \end{cases}$$

KTH

Matrix Functions

Similarly, the notation $G \in \mathbf{L}_1^{m\times n}[0,\infty)$ means that

$$G(t) = \begin{bmatrix} G_{11}(t) & \dots & G_{1n}(t) \\ \vdots & & \\ G_{m1}(t) & \dots & G_{mn}(t) \end{bmatrix} \in \mathbf{R}^{m\times n}, \, t \in [0,\infty)$$

$$\int_0^\infty |G_{ik}(t)| dt < \infty, \quad \forall i,k$$

KTH

Norms, etc.

For $x, y \in \mathbf{L}_2^n[0,\infty)$, define *norm* and *scalar product* by

$$|x| = \left(\int_0^\infty x(t)^T x(t) dt \right)^{1/2}$$

$$< x, y > = \int_0^\infty x(t)^T y(t) dt$$

For $G \in \mathbf{L}_1^{m\times n}[0,\infty)$, $x \in \mathbf{L}_2^n[0,\infty)$, define the *convolution* $G * x$ by

$$(G * x)(t) = \int_0^t G(t-\tau) x(\tau) d\tau$$

Then $(G * x) \in \mathbf{L}_2^m[0,\infty)$.

KTH

Laplace Transform

For a function x on $[0,\infty)$ introduce

$$\mathbf{x}(s) = (\mathcal{L}x)(s) := \int_0^\infty x(t) e^{-st} dt, \quad \mathrm{Re}\, s \geq \sigma$$

If $x \in \mathbf{L}_1^n[0,\infty)$, then integral finite for $\sigma \leq 0$.

Properties

$$\begin{aligned}
(\mathcal{L}\dot{x})(s) &= s\mathbf{x}(s) - x(0) \\
\mathcal{L}\left[\int_0^t x(\sigma)d\sigma\right] &= \frac{1}{s}\mathbf{x}(s) \\
[\mathcal{L}(G * x)](s) &= \mathbf{G}(s)\mathbf{x}(s) \\
\lim_{t\to 0} G(t) &= \lim_{s\to\infty} s\mathbf{G}(s) \\
\lim_{t\to \infty} G(t) &= \lim_{s\to 0} s\mathbf{G}(s) \\
\int_0^\infty x(t)^T y(t)dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{x}(j\omega)^* \mathbf{y}(j\omega)d\omega
\end{aligned}$$

Linear Operator

A *linear operator* is a map $X \to Y$ that satisfies

$$M(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1(Mx_1) + \alpha_2(Mx_2)$$

for $\alpha_1, \alpha_2 \in \mathbf{R}$.

A *bounded* linear operator is one that has a bounded (induced) *norm*

$$\|M\| \quad := \quad \sup_{x\neq 0}\frac{|Mx|}{|x|} = \sup_{|x|=1}|Mx|$$

Example: Matrix operator

Let $M \in \mathbf{R}^{m\times n}, x \in \mathbf{R}^n$. Then $Mx \in \mathbf{R}^m$ and

$$\begin{aligned}
\|M\|^2 &= \sup_{|x|=1}|Mx|^2 \\
&= \sup_{|x|=1} x^T M^T M x \\
&= \max_{1\le i\le n}\lambda_i(M^T M) \\
&= \max_{1\le i\le n}\sigma_i(M)^2
\end{aligned}$$

Example: Convolution operator

Let $G \in \mathbf{L}_1^{m\times n}[0,\infty)$, $x \in \mathbf{L}_2^n[0,\infty)$ and $Mx := G * x \in \mathbf{L}_2^m[0,\infty)$. Then

$$\begin{aligned}
\|M\|^2 &= \sup_{|x|=1}|Mx|^2 = \sup_{|x|=1}\int_0^\infty |G * x|^2 dt \\
&= \sup_{|x|=1}\int_{-\infty}^{\infty} |(\mathcal{L}(G * x))(j\omega)|^2 d\omega \\
&= \sup_{x\neq 0}\frac{\int_{-\infty}^{\infty}|\mathbf{G}(j\omega)\mathbf{x}(j\omega)|^2 d\omega}{\int_{-\infty}^{\infty}|\mathbf{x}(j\omega)|^2 d\omega} = \sup_{\omega\in\mathbf{R}}\|\mathbf{G}(j\omega)\|^2
\end{aligned}$$

This norm is called the H_∞-norm of $\mathbf{G}$:

$$\|\mathbf{G}\|_\infty \quad = \quad \sup_\omega \|\mathbf{G}(j\omega)\| = \sup_\omega \max_i \sigma_i(\mathbf{G}(j\omega))$$

Range and Null Spaces

Range space (Image) of $M : X \to Y$:

$$\mathcal{R}(M) \quad = \quad \{Mx : x \in X\} \subset Y$$

Null space (Kernal) of $M : X \to Y$:

$$\mathcal{N}(M) \quad = \quad \{x : Mx = 0\} \subset X$$

Example

$$\mathcal{R}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) \quad = \quad \left\{\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} : \alpha \in \mathbf{R}\right\}$$

$$\mathcal{N}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) \quad = \quad \left\{\alpha \begin{bmatrix} 2 \\ -1 \end{bmatrix} : \alpha \in \mathbf{R}\right\}$$

Block Diagrams

Diagram for Linearized Satellite

Next Week

- Existence and Uniqueness of Solutions
- Grönwalls Lemma
- Transition Matrices
- Adjoint Operators