TSRT09 – Control Theory

Lecture 9: Circle criterion and describing function

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Summary of lecture 8

Nonlinear systems:

- Superposition principle does not hold

Stability

1. Linearization
   - Linearize system around x_o
   - If linearization is asymptotically stable $\implies$ nonlinear system asymptotically stable near x_o
2. Lyapunov function can be used to show stability

Lecture 9

Stability

- Circle criterion: generalization of Nyquist criterion

Oscillations

- Describing function.

In the book: Ch. 12.3 and 14

Static nonlinearities

Common examples of static nonlinearities:

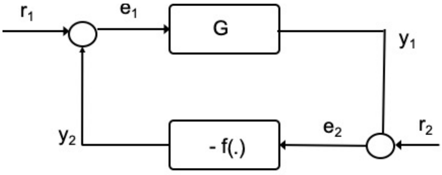
- *Saturations*: ubiquitous; becomes a problem especially if the controller hits hard on the boundaries....
- *Relay*: ideal, with dead zone, with hysteresis,...
- *Deadzone*: vehicle driveline, gear box of industrial robots...
- Others: *backlash*, *hysteresis*,....

System with static (sector) nonlinearity

Nonlinearity $f(\cdot)$

- scalar
- static
- sector nonlinearity

$$k_1 x \le f(x) \le k_2 x, \qquad f(0) = 0$$



Aim: show input-output stability via small gain theorem

Inverse circle criterion

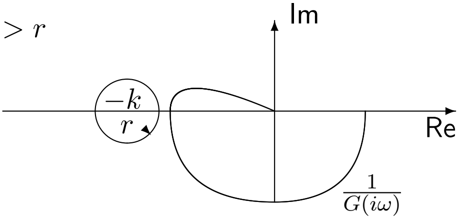
Consider $\tilde{G} = \frac{G}{1+kG}$, $\quad f(y) = \tilde{f}(y) + ky$ and $\quad r = (k_2 - k_1)/2$

Small gain theorem for $\tilde{G}$, $\tilde{f}$ gives

$$r|\tilde{G}(i\omega)| < 1 \quad \iff \quad \frac{1}{|\tilde{G}(i\omega)|} > r$$

or

$$\left|\frac{1}{G(i\omega)} + k\right| > r$$



Transf. $z \to 1/z$ maps inverse circle criterion to the usual one

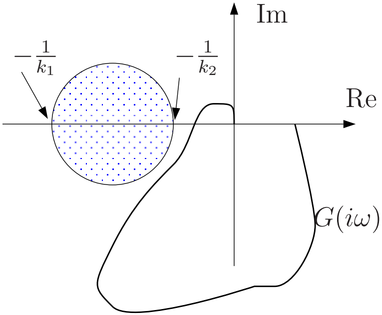
Circle criterion

Linear system $G(s)$ feedback coupled with a static nonlinearity $f(x)$ s.t.

$$f(0) = 0, \quad k_1 \leq \frac{f(x)}{x} \leq k_2$$

The system is stable if

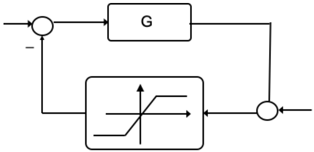
1. $G(s)$ is stable
2. the Nyquist curve of $G(i\omega)$ does not encircle or goes inside the circle

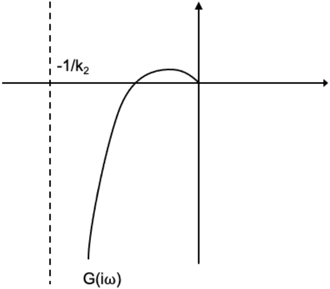


Example: circle criterion for saturation

Example:

$$G(s) = \frac{1}{s(s+1)(s+2)}$$





Describing functions

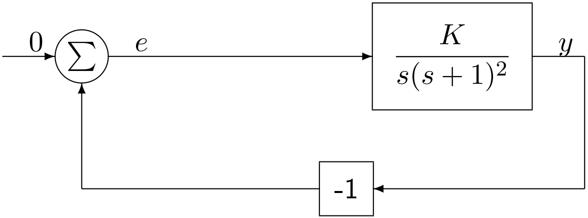
Self-sustained oscillations: oscillations not induced by a periodic input or reference signal

- When does an oscillation occur in a nonlinear system?
- Which amplitude and frequency does it have?
- Is it stable in amplitude?

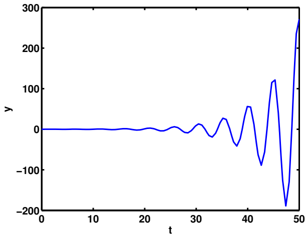
Method: Describe a nonlinearity as an amplitude-dependent gain.

A simple linear feedback system

Example

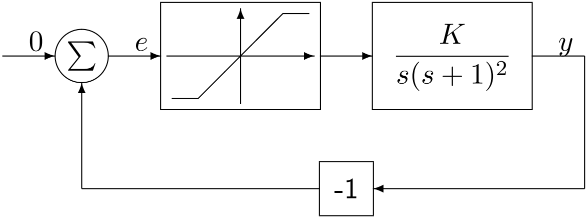


- Unstable for $K = 4$
- Oscillation is growing unbounded

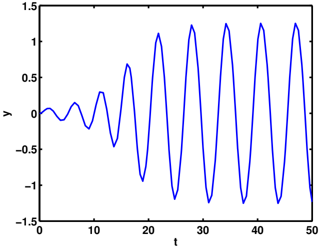


A simple feedback system with saturation

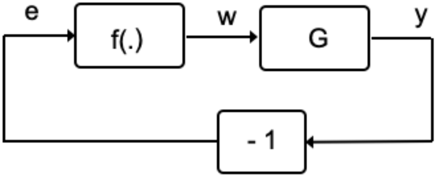
Same system, but now with a saturation in the loop



- Still $K = 4$
- Oscillation is no longer growing unbounded



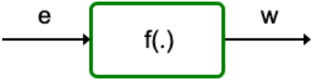
Feedback system with static nonlinearity



- Assume $f(\cdot)$ is a static nonlinearity
- Assume the system exhibits self-sustained oscillations
- **Task**: describe amplitude and frequency of the oscillations

Sine signal through static nonlinearity

$e = C \sin \omega t$ $\qquad$ $w = f(C \sin \omega t)$



Fourier series expansion w:

$$w = f_0(C) + A(C) \sin(\omega t + \phi(C)) + A_2(C) \sin(2\omega t + \phi_2(C)) + A_3(C) \sin(3\omega t + \phi_3(C)) + \dots$$

Fundamental component for the nonlinearity:

- amplitude $A(C)$
- phase $\phi(C)$

Sine through nonlinearity followed by a linear system

$$e = C\sin\omega t$$

$$w = f_0(C) + A(C)\sin(\omega t + \phi(C)) + A_2(C)\sin(2\omega t + \phi_2(C)) + \dots$$

$$y = f_0(C)\underbrace{|G(0)|}_{\text{lin. sys.}} + A(C)\underbrace{|G(i\omega)|}_{\text{lin. sys.}}\sin(\omega t + \phi(C) + \underbrace{\psi(\omega)}_{\text{lin. sys.}})$$
$$+ A_2(C)\underbrace{|G(2i\omega)|}_{\text{lin. sys.}}\sin(2\omega t + \phi_2(C) + \underbrace{\psi(2\omega)}_{\text{lin. sys.}}) + \dots$$

where $\psi(\omega) = \arg G(i\omega)$

Sine through nonlinearity followed by a linear system

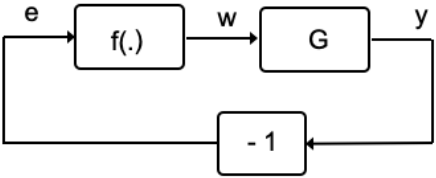
Assumptions:

- $f_0 = 0$ (holds e.g. if f is an odd function)
- $|G(i\omega)| \gg |G(ik\omega)|, \quad k > 1$, i.e., G is a "steep" low-pass filter

Then only the fundamental component matters

$$y \approx A(C)|G(i\omega)| \sin(\omega t + \phi(C) + \psi(\omega))$$

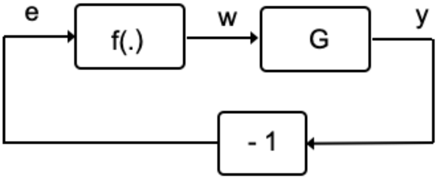
Follow the sine term through a feedback loop



- Neglect everything except the fundamental component
- Condition for oscillations:

$$
\begin{aligned}
e &= C \sin \omega t \\
w &= A(C) \sin(\omega t + \phi(C)) \\
y &= A(C)|G(i\omega)| \sin(\omega t + \phi(C) + \psi(\omega)) \\
e &= -y
\end{aligned}
$$

Follow the sine term through a feedback loop



Conditions for oscillation:

$$A(C)|G(i\omega)| \sin(\omega t + \phi(C) + \psi(\omega)) \; = \; C \sin(\omega t + \pi)$$

- Amplitude: $A(C)|G(i\omega)| = C$
- Phase: $\phi(C) + \psi(\omega) = \pi + 2k\pi$

Describing function

- Compact description of nonlinearity: describing function

$$Y_f(C) = \frac{A(C)e^{i\phi(C)}}{C}$$

- polar form of the fundamental component of $f(\cdot)$ at freq. ω
- Interpretation: "transfer function" of a nonlinearity for a stationary sinusoidal fundamental component
- Gain of $f(\cdot)$ is given by $|Y_f(C)|$ and phase shift by $\arg Y_f(C)$

Describing function: alternative formulation

Y_f can be written equivalently

$$Y_f(C) = \frac{b(C) + ia(C)}{C}$$

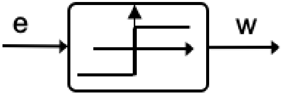
where

$$a(C) = \frac{1}{\pi}\int_0^{2\pi} f(C\sin\alpha)\cos\alpha d\alpha$$
$$b(C) = \frac{1}{\pi}\int_0^{2\pi} f(C\sin\alpha)\sin\alpha d\alpha$$

i.e., the Fourier coefficients associated to the fundamental component (with period normalized by 2π).

Example: ideal relay

$$f(e) = \begin{cases} 1 & \text{if } e > 0 \\ -1 & \text{if } e < 0 \end{cases}$$



gives

$$a(C) = 0$$
$$b(C) = \frac{1}{\pi}\int_0^{\pi} \sin\alpha d\alpha + \frac{1}{\pi}\int_{\pi}^{2\pi} (-\sin\alpha) d\alpha = \frac{4}{\pi}$$

which gives

$$Y_f(C) = \frac{4}{\pi C}$$

Describing function: usage

- $Y_f(C)$ gives a compact description of condition for oscillations: phase and amplitude in a *single* equation

$$\begin{cases} A(C)|G(i\omega)| & = C \\ \phi(C) + \psi(\omega) & = \pi + 2k\pi \end{cases} \iff \quad Y_f(C)G(i\omega) = -1$$

- to see this: linear system in polar form: $G(i\omega) = |G(i\omega)|e^{i\psi(\omega)}$

- Nyquist-like condition: "curve passing through -1"

Calculation of oscillation: graphical method

Equation

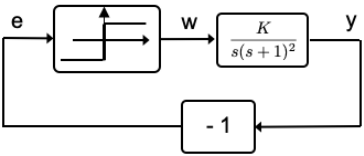
$$Y_f(C)G(i\omega) = -1$$

can be written as

$$-\frac{1}{Y_f(C)} = G(i\omega)$$

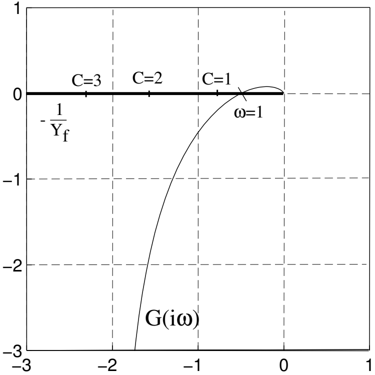
- If $-1/Y_f(C)$ and Nyquist curve $G(i\omega)$ are plotted in the complex plane, the condition for oscillation is that the curves intersect each other
- The intersection gives ω and C for the oscillation

Example: ideal relay in closed loop



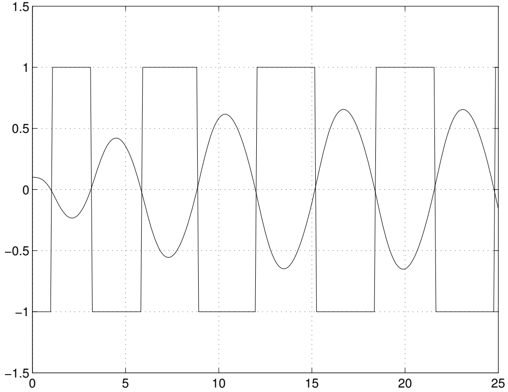
$$Y_f(C) = \frac{4}{\pi C}$$

$$\Longrightarrow \quad -\frac{1}{Y_f(C)} = -\frac{\pi C}{4}$$



$G(i\omega)$ intersect $-1/Y_f(C)$ at $\omega \approx 1$ and $C \approx 0.6$.

Example: ideal relay in closed loop, simulation



square wave $= w$
sinusoid $= y$

Describing function: usage

- What? Method for verifying the existence of self-sustained oscillations in a feedback system with a static nonlinearity
- How? Two steps:
  1. Compute $Y_f(C)$ given the nonlinearity $f(\cdot)$
  2. Solve either

$$\begin{cases} A(C)|G(i\omega)| = C \\ \phi(C) + \psi(\omega) = \pi + \nu 2\pi \end{cases}$$

     or

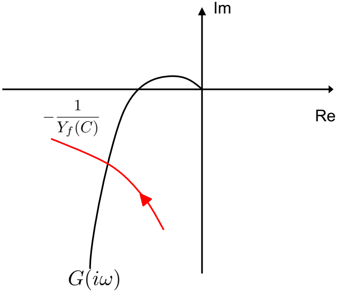
$$G(i\omega) = -\frac{1}{Y_f(C)}$$

- Equation can be solved algebraically or graphically
- It is an *approximated* method, since one makes the assumption that higher frequency components are negligible

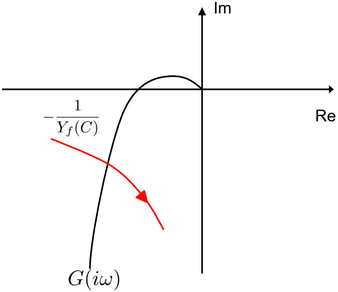
Amplitude stability of oscillations

What happens if the oscillation is perturbed (e.g. disturbance)?

- Intersection $-\frac{1}{Y_f(C)} = G(i\omega)$ can be thought as the point -1 of the Nyquist plot of a linear system $\implies$ "critical point"
- The direction of the arrow indicates how the value of $-1/Y_f(C)$ changes when the amplitude C *grows*



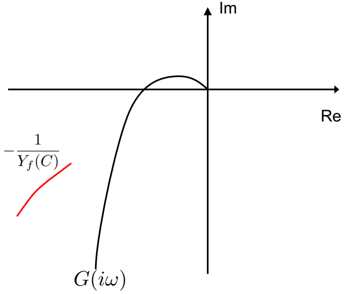
Stable oscillation



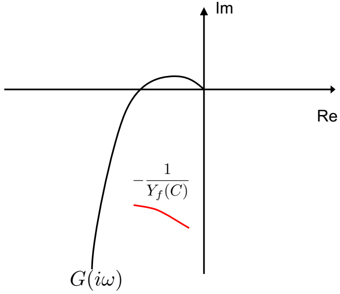
Unstable oscillation

Amplitude stability of oscillations

What happens if $G(i\omega)$ and $-1/Y_f(C)$ do not intersect each other?



Vanishing oscillation



Unbounded growing oscillations

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