TSRT09 – Control Theory

Lecture 8: Nonlinearity and stability

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Schema

w4

w5

w6

w7

TODAY w8

02-24/26: LAB1 w9

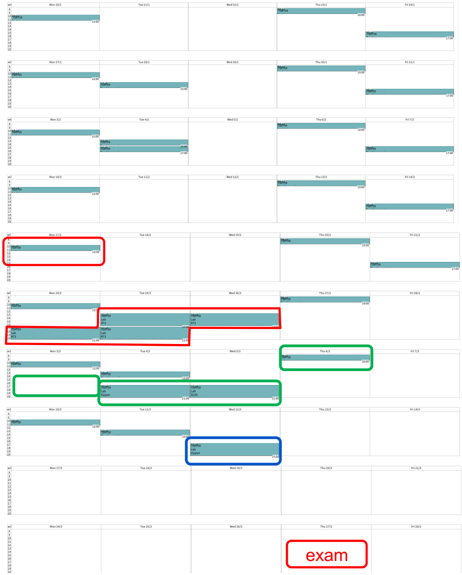
03-03: LAB2 prep eval. w10
03-04/05: LAB2
03-06: LAB2 eval.

w11

03-12: LAB3

w12

03-27: exam w13



Summary of lecture 7

Loop shaping: $\mathcal{H}_2$- and $\mathcal{H}_\infty$- synthesis

- Describe specifications using W_S, W_T and W_u
- Task: make $W_S S$, $W_T T$, $W_u G_{wu}$ "as small as possible"
- $\mathcal{H}_2$: Minimize $\int (|W_S S|_2^2 + |W_T T|_2^2 + |W_u G_{wu}|_2^2)\, d\omega$
- $\mathcal{H}_\infty$: Set absolute constraints for $|W_S S|$, $|W_T T|$, $|W_u G_{wu}|\ \forall\, \omega$
- Both lead to Algebraic Riccati Equation (ARE)

Summary PART II: Linear multivariable regulator synthesis

Control design: summary

- Do an RGA analysis
- Use simple single-loop regulators of PID type if RGA shows that it is possible
- Use other techniques:
  - linear quadratic
  - $\mathcal{H}_2/\mathcal{H}_\infty$-synthesis

PART III: NONLINEAR CONTROL THEORY

- Lecture 8: Nonlinearity and stability
- Lecture 9: Circle criterion and describing function method
- Lecture 10: Phase plane
- Lecture 11: Regulator synthesis and exact linearization

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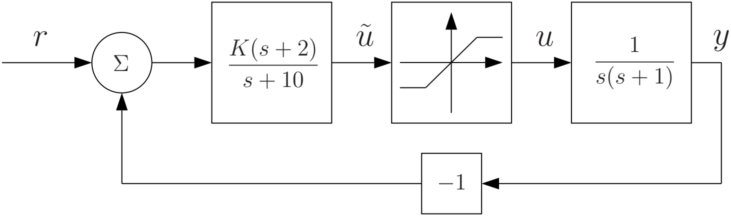
In the book: Ch. 11 and 12

What makes a system (non)linear

Example: DC-motor with control saturation



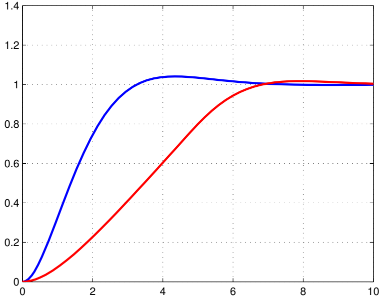
- DC-motor controlled with a lead-lag regulator.
- Task is to control the angle y of the motor.
- Power amplifier driving u is limited $\implies$ "saturation".
- Saturation makes the system nonlinear.

Example: DC-motor with control saturation (cont'd)

DC-motor with angular reference steps of different amplitude.

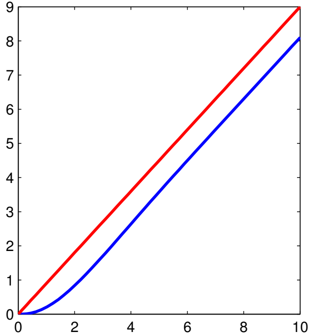
blue: step of amplitude 1
red: step of amplitude 5 (rescaled by $1/5$)

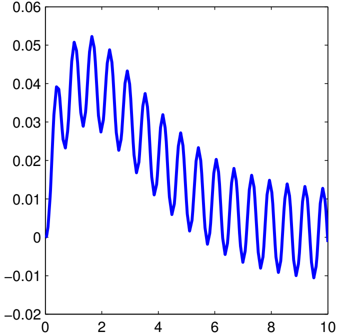
The step response is amplitude-dependent. If the system had been linear then the two curves would have coincided $\rightarrow$ nonlinear



DC-motor (cont'd): ramp and sine responses

red: reference signal r, blue: output y



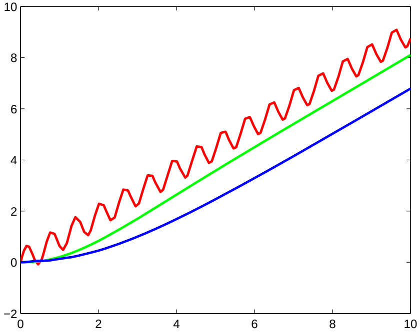


Ramp response:
approximately same as linear

Sine response:
approximately same as linear

DC-motor (cont'd): ramp + sine responses

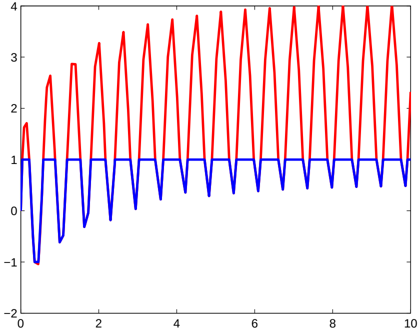
red: r, green: y (when r is a pure ramp – previous slide)
blue: y (when r is a ramp + sine)



Here something happens: sine is not visible and the ramp error has increased... $\Longrightarrow$ no additivity; no frequency fidelity

DC-motor: control signal before and after the saturation

red: before the saturation ($\tilde{u}$). blue: after the saturation (u).



DC-motor (cont'd): conclusions

- Summary:
    - The qualitative appearance of the step response depends on the amplitude $\rightarrow$ not invariant to scaling
    - The effect of different inputs is not additive
    - No frequence fidelity
- Superposition principle does not hold.

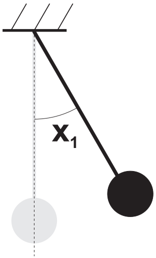
Fairly common nonlinear system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u - ax_2 - b\sin x_1$$

$x_1 =$ angular (phase) position
$x_2 =$ angular velocity
$u =$ external driving (torque)

- Simplified model of a generator (hydraulic, nuclear power, wind power, ...)
- Model of phase locking circuit (frequency and phase modulation, generation of stabilized frequency,...)
- Model of a pendulum

Example: pendulum



Newton law:

$$m\ell\ddot{\theta} + f\ell\dot{\theta} + mg\sin\theta = 0$$

States:

- $x_1 = \theta$ angle
- $x_2 = \dot{\theta}$ angular velocity

Example: Generator connected to the power grid

$$J\ddot{\theta} = M_d - f\dot{\theta} + K\sin(\omega_0 t - \theta)$$

- $\theta =$ angle of rotation of the generator
- $M_d =$ driving torque
- $-f\dot{\theta} =$ damping (friction etc.)
- $K\sin(\omega_0 t - \theta) =$ interaction with the grid
  - Rest of the power grid "rotates" at an angular velocity ω_0
  - The sign of $\theta - \omega_0 t$ determines if generator gives or takes power from the grid

Example: Generator connected to the power grid

Variables:

$x_1 =$ angular position (phase error against the grid) $= \theta - \omega_0 t$

$x_2 =$ angular velocity (derivative of the phase error) $= \dot{\theta} - \omega_0$

Can be expressed in state space form:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{M_d - f\omega_0}{J} - \frac{f}{J}x_2 - \frac{K}{J}\sin(x_1) = u - ax_2 - b\sin(x_1)$$

General nonlinear system; equilibrium point

- Nonlinear system

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t))$$

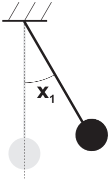
- Equilibrium point:

$$(x_0, u_0) \quad \text{s. t.} \quad f(x_0, u_0) = 0$$

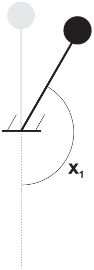
- meaning:

$$\begin{cases} x(0) & = x_0 \\ u(t) & = u_0 \end{cases} \quad \Longrightarrow \quad x(t) = x_0 \ \ \forall t \geq 0$$

Equilibrium point: example (pendulum)



$$x_{0,1} = 0 \pm 2\pi k, \quad , \quad u_0 = 0$$
$$x_{0,2} = 0$$



$$x_{0,1} = \pi \pm 2\pi k, \quad , \quad u_0 = 0$$
$$x_{0,2} = 0$$

$\Longrightarrow$ multiple isolated equilibrium point

Stability

- Stability: definitions
- Stability via linearization
- Stability via Lyapunov function
- Circle criterion

Stability

An equilibrium point x_0 is

- (locally) *(marginally) stable* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x(0) - x_0| \leq \delta \implies |x(t) - x_0| \leq \epsilon \;\; t \geq 0$$

- (locally) *asymptotically stable* if it is stable and there exists a $\delta > 0$ such that

$$x(t) \to x_0, \qquad t \to \infty$$

  whenever $|x(0) - x_0| < \delta$

- *unstable* if it is not (locally) stable

- *globally asymptotically stable* if the δ mentioned above can be taken arbitrarily big

Linearization

- Nonlinear system

$$\dot{x} = f(x, u)$$
$$y = h(x)$$

- Equilibrium point $(x_0,\ u_0)$:

$$0 = f(x_0, u_0)$$
$$y_0 = h(x_0)$$

- Linearization:

$$\frac{d}{dt}(x - x_0) = A(x - x_0) + B(u - u_0)$$
$$y - y_0 = C(x - x_0)$$

where

$$A = \frac{\partial f(x_0, u_0)}{\partial x}, \quad B = \frac{\partial f(x_0, u_0)}{\partial u}, \quad C = \frac{\partial h(x_0)}{\partial x}$$

Stability by linearization

Theorem: Consider a nonlinear system $\dot{x} = f(x, u)$ and its linearization at an equilibrium point (x_0, u_0)

1. If the linearized system is asymptotically stable then so is the original nonlinear system locally, in a neighborhood of (x_0, u_0)
2. If the linearized system is unstable then so is the original nonlinear system
3. If the linearized system has eigenvalues on the imaginary axes (and maybe in the left half of the complex plane) then nothing can be said of the stability character of the nonlinear system

Lyapunov function

- Consider a system

$$\dot{x} = f(x)$$

with an equilibrium x_0

- If there exists a Lyapunov function i.e., a function V s.t. the following conditions are valid in a neighborhood of x_0:

$$V(x) > 0,\ x \neq x_0 \quad V(x_0) = 0, \qquad \text{(i.e., } V(x) \text{ pos. def.)}$$

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x) < 0,\ x \neq x_0 \qquad \text{(i.e., } \dot{V}(x) \text{ neg. def.)}$$

$$\dot{V}(x_0) = 0$$

then the system is locally asymptotically stable at x_0

Lyapunov function

- Global condition: If conditions on V are valid everywhere and

  $V(x) \to \infty \quad |x - x_0| \to \infty$ (i.e., $V(x)$ radially unbounded)

  then x_0 is globally asymptotically stable

Lyapunov function

- Weaker condition: It is enough that

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x) \leq 0 \qquad \text{(i.e., } \dot{V}(x) \text{ neg. semidef.)}$$

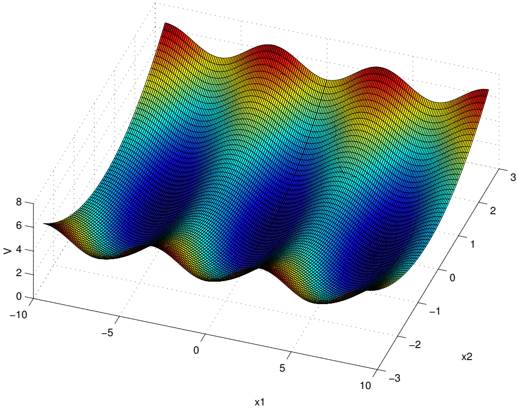
and no solution $x(t)$ (except equilibrium solution $x(t) = x_0$) stays *completely* in the "level surface" $\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x) = 0$

Example: pendulum

Example: pendulum

Example: Lyapunov function for pendulum

$$V(x) = \frac{1}{2}x_2^2 + b(1 - \cos x_1)$$



Lyapunov equation for linear system

Theorem Consider the system $\dot{x} = Ax$, and the equilibrium point $x_o = 0$. The following are equivalent

1. The system is asymptotically stable
2. $\mathrm{Re}\,[\lambda(A)] < 0$ for all $\lambda(A)$
3. For every $Q = Q^T > 0$ there exists a unique $P = P^T > 0$ that solves the Lyapunov equation

$$A^T P + PA = -Q$$

and $V(x) = x^T P x$ is a Lyapunov function for the system

Lyapunov equation for linear system

Lemma 12.1 Assume that the system $\dot{x} = Ax$ is asymptotically stable, that is, all eigenvalues of A lie strictly in LHP. Then for every positive semidefinite matrix Q there exists a positive semidefinite matrix P which solves the equation

$$A^T P + PA = -Q \tag{1}$$

If Q is positive definite, then also P is positive definite.

Conversely, if there are positive semidefinite matrices P and Q such that (1) holds and the pair (A, Q) is detectable, then A has all eigenvalues with negative real part.

Lyapunov equation for almost linear system

- Consider

$$\dot{x} = Ax + g(x)$$

  where A has eigenvalues in LHP and $g(x)$ contains second or higher order term

- Construct a Lyapunov function $V = x^T Px$ for $\dot{x} = Ax$ by solving

$$A^T P + PA = -Q, \quad Q > 0, \; P > 0$$

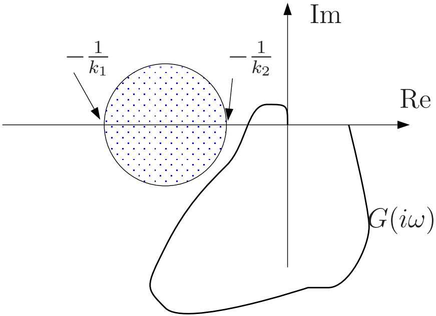
- This V is a Lyapunov function also if the term $g(x)$ is included, provided that x is small enough

Circle criterion

Linear system $G(s)$ is feedback coupled with a static nonlinearity $f(x)$

$$f(0) = 0, \quad k_1 \leq \frac{f(x)}{x} \leq k_2$$

Stability if the Nyquist diagram of $G(i\omega)$ does not encircle and goes inside the circle



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