TSRT09 – Control Theory

Lecture 7: Loop shaping

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Half course questionnaire (12 participants)

- What is the sensitivity function S used for?
  - ☐ design the feedback gain F_y (1)
  - ☐ estimate the impulse response
  - ☐ measure the impact of a disturbance on the system (12)
  - ☐ robustness to model errors (1)
  - ☐ don't know

- If $u(t)$ is a white noise, what is its spectrum $\Phi_u(\omega)$?
  - ☐ impossible to say without further information
  - ☐ a constant (10)
  - ☐ another white noise (1)
  - ☐ a delta function
  - ☐ don't know (1)

Half course questionnaire (12 participants)

- For the MIMO transfer function $G(s)$:
  - ☐ Singular values of G give the poles of G (3)
  - ☐ Singular values of G replace Bode plots (4)
  - ☐ The poles of G can only be computed via a state space realization (3)
  - ☐ u and y can be decoupled if and only if G is diagonal (2)

- True or false?
  - RGA requires solving a ARE ☐ true (1) ☐ false (9)
  - Computing the Kalman filter requires detectability of (A, C) ☐ true (8) ☐ false (2)
  - Controllability implies detectability ☐ true (3) ☐ false (8)
  - Controllability implies stabilizability ☐ true (7) ☐ false (4)
  - Controllability implies arbitrarily pole placement in a state feedback design ☐ true (5) ☐ false (5)

Half course questionnaire (12 participants)

- So far the lectures have been:
  - ☐ too slow
  - ☐ too fast (8)
  - ☐ not very clear
  - ☐ understandable (7)
  - ☐ ... more writing on whiteboard; quite good

- So far the exercise sessions have been:
  - ☐ useful to understand the course content (7)
  - ☐ not so useful (1)
  - ☐ clear (4)
  - ☐ not very clear (1)
  - ☐ I am not attending them (2)
  - ☐ ...

Half course questionnaire (12 participants)

- For the following topics mentioned in class, my background is
  - Linear algebra (norms, matrix properties, etc.)

    ☐ adequate (8) ☐ so and so (4) ☐ inadequate

  - Control and systems:

    ☐ adequate (8) ☐ so and so (4) ☐ inadequate

  - Statistics/probability:

    ☐ adequate (9) ☐ so and so (3) ☐ inadequate

- What is the argument that you have found most confusing?
  - Singular values and SVD;
  - Stochastic things
  - RGA and decoupling
  - Problem is the time to process each lecture before next one

Half course questionnaire (12 participants)

- In the lectures, I would have preferred:
  - ☐ more in-depth explanations
  - ☐ less theory, more examples (6)
  - ☐ earlier start for the labs (3)
  - ☐ lectures in swedish (1)
  - ☐ slower tempo;
  - ☐ less theory would not be possible, but more examples would be nice

- Any comment you want to add?
  - More structured writing on whiteboard (e.g., with titles)
  - Fun course and content
  - Interesting course

Summary of lecture 6

LQG problem

$$\begin{aligned} \text{Minimize} \quad & E[x^T \bar{Q}_1 x + u^T Q_2 u] \\ \text{s.t.} \quad & \dot{x} + Ax + Bu + Nv_1 \\ & z = Mx \\ & y = Cx + v_2 \end{aligned}$$

This gives the regulator

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K(y - C\hat{x}) \\ u &= -L\hat{x} + L_r r \\ L &= Q_2^{-1} B^T S, \quad K = PC^T R_2^{-1} \end{aligned}$$

where S and P are obtained from the algebraic Riccati equation

$$A^T S + SA + \bar{Q}_1 - SBQ_2^{-1} B^T S = 0$$

$$AP + PA^T + NR_1 N^T - PC^T R_2^{-1} CP = 0$$

Summary of lecture 6 (cont'd)

Robustness:

- Full state LQ feedback: $u = -Lx$:
  - if $G_o = GF_y$ is the loop gain

$$|I + G_o(i\omega))| \geq 1$$

  - Highly robust and insensitive:
    1. infinite amplitude margin
    2. at least 60° of phase margin
    3. $|S(i\omega)| \leq 1, \ |T(i\omega)| \leq 2$

- LQG (with Kalman filter)
  - No guaranteed robustness, nor (in)sensitivity
  - LTR ("Loop Transfer Recovery"): Modification of Kalman filter so as to approach the sensitivity one would get with a full state feedback

Lecture 7

- Frequency-domain specifications for the closed-loop system
- An extended system that include specifications
- $\mathcal{H}_2$ control design
- $\mathcal{H}_\infty$ control design

In the book: Ch. 6.6 and 10

Pros and cons of linear quadratic synthesis

(+) Under stabilizability and detectability, all reasonable choices of Q_1, Q_2, R_1, R_2 ($Q_1 \geq 0$, $Q_2 > 0$, $R_1 \geq 0$, $R_2 > 0$) give a closed-loop system with poles strictly in LHP

(+) Handles well the trade-off between choice of magnitude of different components of x och u

(+) Easy to adjust Q_1, Q_2 so as to get good responses in the time domain

(+) Some possibilities of handling robustness

(-) Difficult to see how Q_1, Q_2, R_1, R_2 influence S, T, G_{wu}, etc. i.e., to include frequency domain specifications

(-) Controller of high dimension

Feedback design with given specifications

Task: Given the system $y = Gu + w$, design $u = -F_y y$ so that the closed-loop system meets the given specifications in S, T, G_{wu}

Specifications

1. Sensitivity function $S = (I + GF_y)^{-1}$ small
   - output disturbance → controlled signal
   - model error → controlled signal
2. Complementary sensitivity function

$$T = (I + GF_y)^{-1} GF_y = I - S \qquad \text{small}$$

   - measurement error → controlled signal
   - model error → stability
3. G_{wu} $G_{wu} = -F_y(I + GF_y)^{-1} = -(I + F_y G)^{-1} F_y$ small
   - output disturbance → input

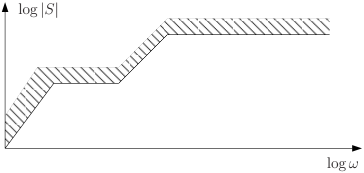
Loop shaping

Loop shaping = "shaping" of the loop gain $GF_y \implies$ direct synthesis of the regulator F_y based on specification on S, T, G_{wu}

- SISO system: classical methods (lead-lag). Intuitive design on the Bode diagram.
- MIMO systems:
  - $\mathcal{H}_2$ control design: Search a compromise through optimization
  - $\mathcal{H}_\infty$ control design: Meet all constraints for S, T, G_{wu}

Example of specification: S

- Often one wants to decrease the sensitivity for low frequencies (e.g. zero steady-state error in step response, etc.).



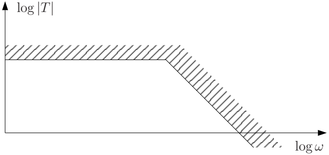
- $W_S(s) =$ weight function for S
- Choose W_S such that

$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|,\ \forall\omega \iff |W_S(i\omega)S(i\omega)| \leq 1\ \forall\omega$$

$$\iff \|W_S S\|_\infty \leq 1$$

Example of specification: T

- Often one want to decrease the complementary sensitivity at high frequencies (measurement noise and model error $|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}$)



- $W_T(s)$ = weight function for T
- Choose W_T such that

$$|T(i\omega)| < |W_T^{-1}(i\omega)|,\ \forall\omega \iff |W_T(i\omega)T(i\omega)| \leq 1\ \forall\omega$$

$$\iff \|W_T T\|_\infty < 1$$

Example of specification: G_{wu}

- $|G_{wu}|$ is the TF gain from system disturbance to the control
- This gain is important to check that the amplitude of the control signal remains moderate
- $W_u(s) =$ weight function for G_{wu}
- Choose W_T such that

$$|G_{wu}(i\omega)| \leq |W_u^{-1}| \; \forall \omega \iff |W_u(i\omega)G_{wu}(i\omega)| \leq 1$$

$$\iff \|W_u G_{wu}\|_\infty \leq 1$$

Specifications

$W_S(i\omega),\ W_T(i\omega),\ W_u(i\omega)$ = weighting of S, T, G_{wu}:

Task: design a controller $u = -F_y y$ so that

$$|W_S(i\omega)S(i\omega)|, \qquad |W_T(i\omega)T(i\omega)|, \qquad |W_u(i\omega)G_{wu}(i\omega)|$$

are all "as small as possible" for all ω

"As small as possible"

1. Mean square sum: $\mathcal{H}_2$-synthesis

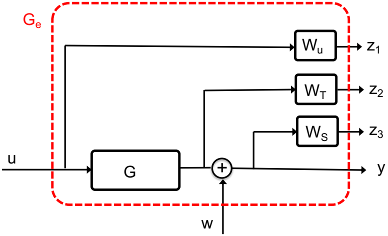
$$\int_{-\infty}^{+\infty} \left(|W_S(i\omega)S(i\omega)|_2^2 + |W_T(i\omega)T(i\omega)|_2^2 + |W_u(i\omega)G_{wu}(i\omega)|_2^2 \right) d\omega$$

2. "Worst case": all less than some γ: $\mathcal{H}_\infty$-synthesis

$$\|W_S S\|_\infty < \gamma, \qquad \|W_T T\|_\infty < \gamma, \qquad \|W_u G_{wu}\|_\infty \leq \gamma$$

To include the specifications: extended system G_e

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & W_u \\ 0 & W_T G \\ W_S & W_S G \\ I & G \end{bmatrix}}_{G_e} \begin{bmatrix} w \\ u \end{bmatrix}$$



$$\begin{bmatrix} z \\ y \end{bmatrix} = [G_e] \begin{bmatrix} w \\ u \end{bmatrix}$$

State space realization of G_e:

$$\dot{x} = Ax + Bu + Nw$$
$$z = Mx + Du$$
$$y = Cx + w$$

A special property of the structure

- State space realization of G_e

$$\dot{x} = Ax + Bu + Nw$$
$$z = Mx + Du$$
$$y = Cx + w$$

is in innovation form, since $v_1 = v_2 = w$ white noise

- If

1. $D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$
2. $A - NC$ has all eigenvalues strictly in the LHP

$\Longrightarrow$ system is its own Kalman filter with

$$K = N$$

Hence the Kalman filter becomes:

$$\dot{\hat{x}} = A\hat{x} + Bu + N(y - C\hat{x}) = A\hat{x} + Bu + Nw$$

Example: DC-servosystem

$$G(s) = \frac{20}{s(s+1)}$$

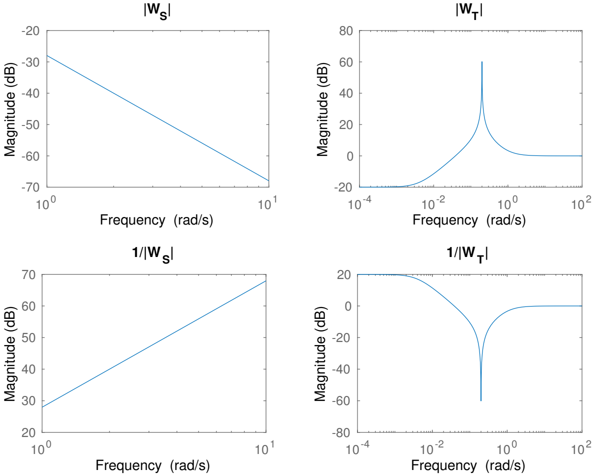
- Specifications:
  - S should decrease to zero with "slope 2" at low frequencies ($\implies$ ramp disturbance gives zero steady state error)
  - T very small at 0.2 rad/s (measurement disturbance / model error).
- Choice of the weight functions (reflects the specifications):

$$W_S(s) = \frac{0.04}{s^2} \quad W_T(s) = \frac{s^2+s+0.04}{s^2+0.001s+0.04}, \quad W_u(s) = 1$$

- For those frequencies at which we want to push down S and T we choose W_S resp. W_T large

Example: DC-servosystem

$$W_S(s) = \frac{0.04}{s^2} \qquad W_T(s) = \frac{s^2 + s + 0.04}{s^2 + 0.001s + 0.04}$$



DC-servosystem, extended model

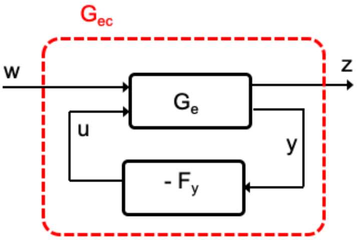
- State space representation of G_e

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.999 & 0 & 0 & -0.001 & 1 \\ 0 & 0 & 0 & 0 & -0.04 & 0 \end{bmatrix} x + \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} w$$

$$z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.04 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} x + w$$

Compact description for the specifications



When the system is feedback regulated with $u = -F_y y$ it holds:

$$z_1 = W_u G_{wu} w$$
$$z_2 = -W_T T w$$
$$z_3 = W_S S w$$

Closed-loop system:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underbrace{\begin{bmatrix} W_u G_{wu} \\ -W_T T \\ W_S S \end{bmatrix}}_{G_{ec}} w \quad \Longleftrightarrow \quad z = G_{ec} w$$

Specifications

$W_S(i\omega),\ W_T(i\omega),\ W_u(i\omega) =$ weighting of S, T, G_{wu}:

Task: design a controller $u = -F_y y$ so that

$$|W_S(i\omega)S(i\omega)|, \qquad |W_T(i\omega)T(i\omega)|, \qquad |W_u(i\omega)G_{wu}(i\omega)|$$

are all "as small as possible" for all ω

"As small as possible"

1. Mean square sum: $\mathcal{H}_2$-synthesis

$$\int_{-\infty}^{+\infty} \left(|W_S(i\omega)S(i\omega)|_2^2 + |W_T(i\omega)T(i\omega)|_2^2 + |W_u(i\omega)G_{wu}(i\omega)|_2^2 \right) d\omega$$

2. "Worst case": all less than some γ: $\mathcal{H}_\infty$-synthesis

$$\|W_S S\|_\infty < \gamma, \qquad \|W_T T\|_\infty < \gamma, \qquad \|W_u G_{wu}\|_\infty \leq \gamma$$

Optimal $\mathcal{H}_2$-regulator

$\mathcal{H}_2$-norm of G_{ec}

$$\begin{aligned}
V(F_y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Big(|W_S(i\omega)S(i\omega)|_2^2 + |W_T(i\omega)T(i\omega)|_2^2 \\
&\qquad\qquad + |W_u(i\omega)G_{wu}(i\omega)|_2^2\Big)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{tr}\left(\left[(W_uG_{wu})^* \ (W_TT)^* \ (W_SS)^*\right]\begin{bmatrix} W_uG_{wu} \\ W_TT \\ W_SS \end{bmatrix}\right)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{tr}\left(G_{ec}^*(i\omega)G_{ec}(i\omega)\right)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}|G_{ec}(i\omega)|_2^2 d\omega \\
&= \|G_{ec}\|_2^2
\end{aligned}$$

Optimal $\mathcal{H}_2$-regulator

Problem: Compute the feedback $u = -F_y y$ that minimizes the $\mathcal{H}_2$-norm

$$V(F_y) = \|G_{ec}\|_2^2 = \|z\|_2^2 = \|Mx\|_2^2 + \|u\|_2^2$$

for the closed-loop extended system G_{ec}:

$$z = G_{ec} w \quad \iff \quad \begin{aligned} \dot{x} &= Ax + Bu + Nw \\ z &= Mx + Du \\ y &= Cx + w \end{aligned} \qquad u = -F_y y$$

Solution: LQG!

$$F_y \; : \; \begin{cases} \dot{\hat{x}} &= A\hat{x} + Bu + N(y - C\hat{x}) \\ u &= -L\hat{x} \end{cases}$$

with $L = B^T S$ where S solves ARE

$$A^T S + SA + M^T M - SBB^T S = 0$$

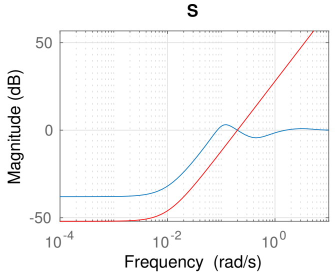
$\mathcal{H}_2$-regulator in practice

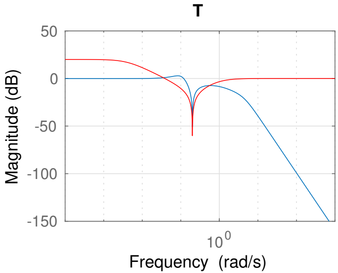
Practical steps:

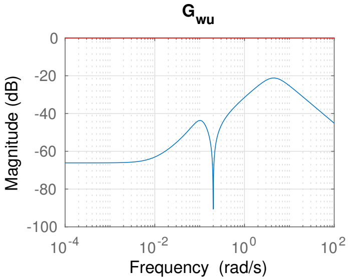
1. Given G
2. Choose W_S, W_T and W_u
3. Compute extended system G_e
4. Compute F_y, i.e., K ($= N$ if possible) and L via LQG
5. Verify your solution:
    - Compute S, T and G_{wu} for G_{ec}
    - Compare them with W_S^{-1}, W_T^{-1} and W_u^{-1}

    If the solution cannot be accepted go to step 2.

Example: $\mathcal{H}_2$ regulation of DC-servosystem







Red: W_*^{-1}, Blue: $\mathcal{H}_2$ design

- S: "Slope 2".
- T: Suppression of measurement noise at 0.2 rad/s.

Optimal $\mathcal{H}_\infty$-regulator

Problem: Find a regulator F_y such that for some γ

$$\begin{aligned} \|G_{ec}\|_\infty \leq \gamma \quad &\iff \quad \|z\|_2 \leq \|G_{ec}\|_\infty \|w\|_2 \\ &\iff \quad \|z\|_2 \leq \gamma \|w\|_2 \\ &\iff \quad V(F_y) = \|z\|_2^2 - \gamma^2 \|w\|_2^2 \leq 0 \end{aligned}$$

This implies

$$\begin{aligned} |W_S(i\omega)S(i\omega)| &\leq \gamma \quad \forall \omega \\ |W_T(i\omega)T(i\omega)| &\leq \gamma \quad \forall \omega \\ |W_u(i\omega)G_{wu}(i\omega)| &\leq \gamma \quad \forall \omega \end{aligned}$$

Optimal $\mathcal{H}_\infty$-regulator

Solution: If

$$A^T S + SA + M^T M - S(BB^T - \gamma^{-2} NN^T)S = 0$$

has a positive semidefinite solution $S = S_\gamma$ which makes $A - BB^T S_\gamma$ stable, then the regulator

$$F_y \ : \ \begin{cases} \dot{\hat{x}} & = A\hat{x} + Bu + N(y - C\hat{x}) \\ u & = -L_\infty \hat{x}, \qquad L_\infty = B^T S_\gamma \end{cases}$$

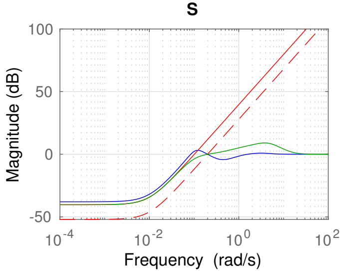
is a solution s.t. $\|G_{ec}\|_\infty < \gamma$

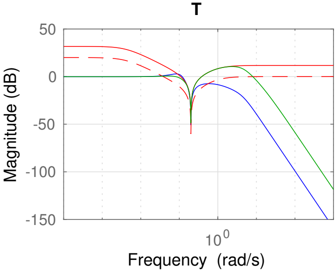
$\mathcal{H}_\infty$ regulator in practice

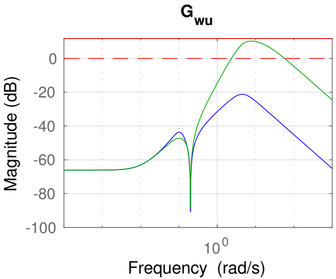
1. Given G
2. Choose W_u, W_S, W_T
3. Build the extended system G_e
4. Choose a γ
5. Solve the LQG problem for the ARE
    - If not solution exists, go to step 4 and increase γ
    - If a solution exists, accept it, or go to step 4 and decrease γ
6. Verify your solution
    - Compute S, T and G_{wu} for G_{ec}
    - Compare them with W_S^{-1}, W_T^{-1} and W_u^{-1}

   If the solution cannot be accepted go to step 2.

Example: $\mathcal{H}_\infty$ regulation of DC-servosystem







- Red, solid: γW_*^{-1}
- Red, dashed: W_*^{-1}
- Blue: $\mathcal{H}_2$ design
- Green: $\mathcal{H}_\infty$ design
- Notice that the constraints "replace" each other!

Pros and cons of $\mathcal{H}_2$, $\mathcal{H}_\infty$

(+) Handles directly frequency domain specifications on S, T, G_{wu}

(+) Tells when the specifications are impossible (via γ)

(+) Easy to weights different specifications (in the frequency domain) against each other

(-) Can be difficult to control the behavior in detail in the time domain

(-) Often gives complicated regulators (number of states in the regulator = total number in G, W_u, W_S, W_T)

Linear multivariable regulator synthesis

Control design: Summary

- Do an RGA analysis
- Use simple single-loop regulators of PID type if RGA shows that it is possible
- Use other linear quadratic or $\mathcal{H}_2/\mathcal{H}_\infty$-synthesis

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Lecture 7

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