TSRT09 – Control Theory

Lecture 6: LQ-regulator

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Summary of lecture 5: RGA, Decoupled control, IMC

- RGA measures the cross-coupling / interaction between inputs and outputs in a MIMO system.
- Decentralized regulation
  - Make a regulator for a MIMO system by letting *one* input be regulated by *one* output
  - Result is a number of independent "single variable loops"
  - RGA can be used for choosing which input should be regulated by which output
- Decoupled control
  - Can be used if "natural pairs" of inputs and outputs are missing.
  - Creates a new "virtual system" which is almost diagonal which can be regulated by means of a decentralized regulator.
- IMC regulator: contains a model G of the regulated system, and the feedback is from $y - Gu$, i.e., from the impact of model error and disturbances.

Summary of lecture 5: LQ with full state feedback

Problem: Find a $u = -Lx$ that solves

$$\begin{aligned} \min J &= \int_0^\infty (z^T Q_1 z + u^T Q_2 u) dt \\ &= \int_0^\infty (x^T \bar{Q}_1 x + u^T Q_2 u) dt \qquad \bar{Q}_1 = M^T Q_1 M \\ \text{s. t. } \dot{x} &= Ax + Bu, \qquad x(0) \text{ given} \\ z &= Mx \end{aligned}$$

Solution: $u = -Lx$, $L = Q_2^{-1} B^T S$, where S solves the ARE

$$A^T S + SA + \bar{Q}_1 - SBQ_2^{-1} B^T S = 0$$

and $A - BQ_2^{-1} B^T S$ is stable

Lecture 6

- LQ/LQG + Kalman filter.
- Setting an LQ/LQG-regulator in practice.
- Robustness of LQ- and LQG-regulators.

In the book: Ch. 9

Similarities between Kalman filter and LQ

Kalman filter:

$$AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP = 0$$
$$K = PC^TR_2^{-1} \qquad (\Leftrightarrow\ K^T = R_2^{-1}CP)$$

LQ-regulator:

$$A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS = 0$$
$$L = Q_2^{-1}B^TS$$

$$A^T \longleftrightarrow A$$
$$C^T \longleftrightarrow B$$
$$N^T \longleftrightarrow M$$
$$R_1 \longleftrightarrow Q_1$$
$$R_2 \longleftrightarrow Q_2$$
$$P \longleftrightarrow S$$
$$K^T \longleftrightarrow L$$

Both problems require solving an ARE

LQ problem: stochastic version

- Model with disturbances and output:

$$\dot{x} = Ax + Bu + Nv_1,$$
$$z = Mx,$$
$$y = Cx + v_2$$

- v_1, v_2 white noise of covariance R_1, R_2.
- Task: seek a linear feedback that minimizes the criterion

$$E[z^T Q_1 z + u^T Q_2 u] = E[x^T \bar{Q}_1 x + u^T Q_2 u], \quad \bar{Q}_1 = M^T Q_1 M$$

- If v_1 and v_2 are Gaussian white noises the method is called Linear Quadratic Gaussian Control (LQG).
- Can be transformed to a "nicer" deterministic problem

LQG: Facts

Fact 1: observer

- x is no longer available for state feedback $\implies$ need an observer
- Kalman filter

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

- In a Kalman filter

  1. $\hat{x}$ and $\tilde{x}$ are uncorrelated: $E[\tilde{x}\hat{x}^T] = 0$
  2. $\tilde{x}$ is independent of u. (See Lemma 5.3)

$\implies$ Separation principle: can split the problem into optimal state estimation + optimal control

$$E[x^T\bar{Q}_1x + u^TQ_2u] = \underbrace{E[\tilde{x}^T\bar{Q}_1\tilde{x}]}_{\text{Kalman filter}} + \underbrace{E[\hat{x}^T\bar{Q}_1\hat{x} + u^TQ_2u]}_{\text{LQ problem}}$$

LQG: Facts

Fact 2: innovation

- In a Kalman filter the innovation

$$\nu = y - C\hat{x}$$

  is a white noise with intensity R_2. (See Thm 5.5)

- $\implies$ Kalman filter can be written in innovation form

$$\dot{\hat{x}} = A\hat{x} + Bu + K\nu$$

  where the noise input ν is white noise, $\Phi_\nu(\omega) = R_2 =$ const

LQG: facts

Fact 3: white noise response $\sim$ impulse input

- call $\xi(t) = R_2^{1/2}\delta(t)$
- $\implies \mathcal{F}[\xi(t)] = R_2^{1/2}$
- $\implies \Phi_\xi(\omega) = \mathcal{F}[\xi(t)]\mathcal{F}^*[\xi(t)] = R_2^{1/2}R_2^{1/2} = R_2$
- $\implies \xi(t) = \begin{cases} \text{w.n. of covariance } R_2 \\ \text{impulse of amplitude } R_2^{1/2} \end{cases}$
- $\implies$ the stochastic problem can be reformulated as a deterministic problem

$$\dot{\hat{x}} = A\hat{x} + Bu + KR_2^{1/2}\delta(t)$$

LQG: facts

Fact 4: impulse input $\sim$ non-zero initial condition

- Adding an impulse to the input of a system is equivalent to adding a certain initial condition when the input is zero

$$\begin{cases} \dot{\hat{x}} & = A\hat{x} + Bu + KR_2^{1/2}\delta(t) \\ \hat{x}(0) & = 0 \end{cases} \iff \begin{cases} \dot{\hat{x}} & = A\hat{x} + Bu \\ \hat{x}(0) & = KR_2^{1/2} \end{cases}$$

- $\implies$ problem can be formulated as an optimal control problem with a given initial state

Consequence of Facts 1–4

$\Longrightarrow$ the stochastic optimization problem is equivalent to minimizing

$$\int_0^\infty (\hat{x}^T \bar{Q}_1 \hat{x} + u^T Q_2 u) dt$$

for the system

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \text{given } \hat{x}(0) = K R_2^{\frac{1}{2}}$$

$\Longrightarrow$ "ordinary" LQ problem with $K =$ Kalman gain

Solution: $u = -L\hat{x}$, $L = Q_2^{-1} B^T S$, where S is given by

$$A^T S + SA + \bar{Q}_1 - SBQ_2^{-1} B^T S = 0$$

Complete LQG (optimal control + Kalman filter)

Problem: Seek a feedback $u = -F_y y$ that solves

$$\min J = \|z\|_{Q_1}^2 + \|u\|_{Q_2}^2 = E[z^T Q_1 z + u^T Q_2 u]$$

$$\text{s. t.} \quad \dot{x} = Ax + Bu + Nv_1$$

$$z = Mx$$

$$y = Cx + v_2$$

where

- penalties $Q_1 = Q_1^T \geq 0, \quad Q_2 = Q_2^T > 0$
- $v_1,\ v_2$ uncorrelated white noise of covariance
  $R_1 = R_1^T \geq 0,\ \ R_2 = R_2^T > 0$

Complete LQG (optimal control + Kalman filter)

Solution: Assume

$(A,\, B)$ stabilizable

$(A,\, C)$ detectable

$(A,\, M^T Q_1 M)$ detectable

$(A,\, N^T R_1 N)$ stabilizable

Optimal controller is

$$u = -L\hat{x}, \qquad L = Q_2^{-1} B^T S$$

where

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

and

$$K = PC^T R_2^{-1}$$

with $S = S^T \geq 0$ solution of the ARE

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0$$

and $P = P^T \geq 0$ solution of the ARE

$$AP + PA^T + NR_1 N^T - PC^T R_2^{-1} CP = 0$$

LQG: How does the optimal regulator look like?

- The complete regulator $(-F_y)$ in state space form:

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$
$$u = -L\hat{x}$$

has dimension n

- Transfer function $y \to u$

$$F_y(s) = L(sI - A + BL + KC)^{-1}K$$

- Sensitivity

$$S = (I+GF_y)^{-1} = \big(I+C(sI-A)^{-1}BL(sI-A+BL+KC)^{-1}K\big)^{-1}$$

Practical use

- Parameters: Q_1, Q_2 (penalty matrices),
  R_1, R_2 (noise covariances)
- "Real" requirements must be translated into values for the penalty matrices
- In practice the penalty matrices are chosen in an iterative process in which one tests different values and evaluate the results
- Often these matrices are chosen diagonal
- What matters is the *ratio* between Q_1 and Q_2, resp. *ratio* between R_1 and R_2, i.e., multiplying *both* Q_1 and Q_2 for instance by 10 the regulator remains (almost) the same
- Normally one works with powers of 10 when changing the weight matrices, to be able to see the essential differences in the closed-loop system.

Practical use

- Big Q_1 (small Q_2) increases the bandwidth of the system, but also the amplitude of the input
- Choosing different diagonal elements in Q_1 and Q_2 one can balance the control accuracy resp. the input strength of the various components against each other
- Big R_1 (small R_2): trust more measurement than model
  $\Longrightarrow$ regulator tuned to cope with big process noise
  $\Longrightarrow$ decreases the sensitivity function
  increases the complementary sensitivity function
- Often one must "color" the system or the measurement disturbance in oder to get good S or T

Example: DC-servosystem

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + e$$

- x_1=angular position, x_2=angular velocity
- $v =$ system noise, $e =$ measurement noise
- System noise enters the system together with the input u
- No system noise in the equation that links position and velocity, which is an “exact” integrator.
- System is controllable and observable

Example: DC-servosystem

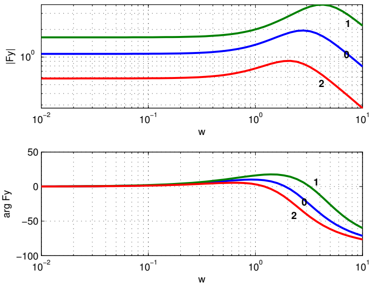
x_1=angular position, x_2=angular velocity, $z = [1\ 0]\, x = x_1$

- Basic regulator (0):
  $\bar{Q}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = 0.1, \quad R_1 = 1, \quad R_2 = 0.1$
- Fast regulator (1): $Q_2 = 0.01$ (smaller penalty to the control signal).
- Slow regulator (2): $R_2 = 1$ (rely less on measurement).

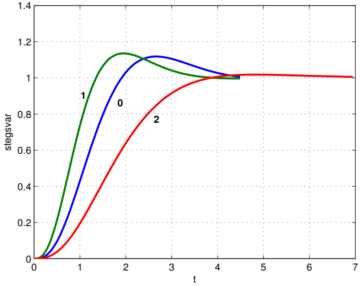
- Poles, basic regulator: $-1.3532 \pm 1.1537i$ (double)
- Poles, fast regulator: $-2.2913 \pm 2.1794i, \quad -1.3532 \pm 1.1537i$.
- Poles, slow regulator: $-1.3532 \pm 1.1537i, \quad -0.8660 \pm 0.5000i$.

DC-servosystem: Regulator (F_y) & step response

0: nominal, 1: smaller Q_2 (fast reg.), 2: bigger R_2 (slow reg.)



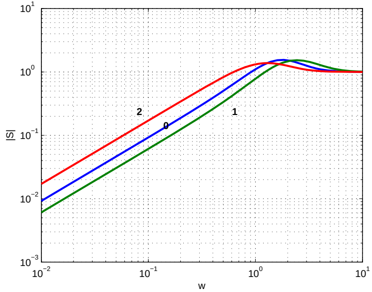
F_y



Step response
(assume $F_r = F_y$)

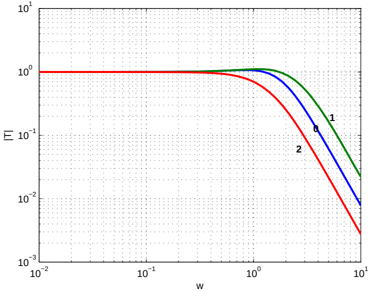
DC-servosystem: S & T

0: nominal, 1: smaller Q_2 (faster), 2: bigger R_2 (slower)



S

(process disturbance $\rightarrow$ controlled signal)



T

(measurement disturbance $\rightarrow$ controlled signal)

DC-servosystem: Low frequency disturbance at the input

Low frequency disturbance modeled as a white noise through e.g. the system

$$v = \frac{0.1}{s + 0.1} w$$

Extended state space description:

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -0.1 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} w$$

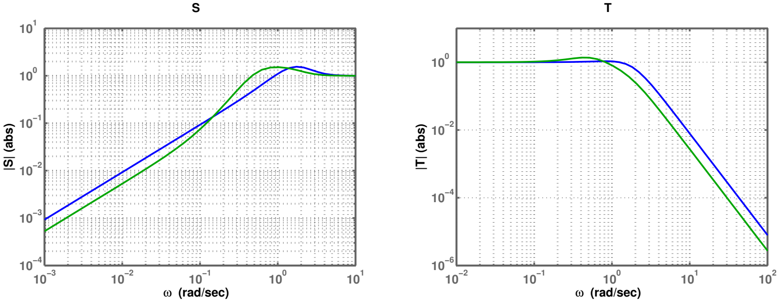
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \bar{x} + e$$

where $w =$ white noise

$\Longrightarrow$ eliminate effects of (quasi) constant disturbances at steady state

DC-servosystem: the noise model shapes S and T

Blue: original model   Green: low frequency disturbance at the input.



- Low frequency process noise pushes down S for low frequencies.
- Lack of high frequency process noise pushes down T for high frequencies
- $\Longrightarrow$ colored noise provides extra freedom for shaping S, T,...

Reference signal

General methodology

- Find out the spectrum of r.
- Describe r as white noise through a linear system.
- Represent this linear system in state space form and insert it into the original state space model i.e., r is considered as a state in the model.
- Represent y and r as measured signals for this system.
- Run LQG-algorithm.
- Result is a feedback from a Kalman filter with r and y as inputs

$$u = F_r\, r - F_y\, y$$

Constant reference signal

- If the reference signal is constant ($\neq 0$) u normally does not have zero mean. It is then natural to modify the criterion to

$$E\left[(z-r)^T Q_1 (z-r) + (u-u^*(r))^T Q_2 (u-u^*(r))\right]$$

where $u^*(r)$ is the input signal which is needed is order to have $z = r$ at steady state

- Result:

$$u = -L\hat{x} + L_r r$$

where L is computed as if $r = 0$ and L_r is given by

$$L_r = (M(BL-A)^{-1}B)^{-1}$$

(if z and u have the same dimension).

Pros & cons of LQG

Cons:

- Even though LQG gives an "optimal" regulator, in practice one must iterate the calculation over different penalty matrices
- Dimension of the regulator F_y can be large
- Difficult to interpret in the frequency domain

Pros:

- One always gets a stable closed-loop system. (Given that $Q_1 \geq 0$, $Q_2 > 0$, $R_1 \geq 0$, $R_2 > 0$, and the system is stabilizable and detectable.)
- MIMO system is as easy to handle as a scalar system
- Easy to interpret in the time domain
- The feedback system uses automatically the measured states (not the estimates), which is a significant robustness against disturbances and model error at the input

Robustness of full state LQ feedback

- Regulator

$$u = -Lx$$

- System transfer functions

$$G(s) = (sI - A)^{-1}B, \qquad F_y = L$$

- Loop gain

$$G_o(s) = L(sI - A)^{-1}B$$

- If $L =$ full-state LQ gain for some A, B, Q_1, Q_2, it holds:

$$(I + G_o(-i\omega))^T Q_2 (I + G_o(i\omega)) \geq Q_2$$

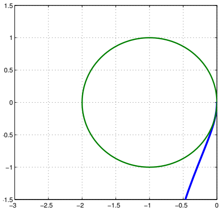
- Consequence:

$$|I + G_o(i\omega)| \geq 1$$

Robustness of full state LQ feedback

SISO case: $|1 + G_o(i\omega)| \geq 1$

- Nyquist diagram interpretation: distance of $G_o(i\omega)$ to -1 is greater or equal to 1
- All Nyquist diagrams for a full state LQ feedback lie *always* outside the green circle



$$\Longrightarrow \begin{cases} \text{Infinite amplitude margin} \\ \text{At least } 60^\circ \text{ phase margin} \end{cases}$$

$\Longrightarrow$ good robustness properties!

MIMO case: similar interpretation, but for singular values

Robustness of full state LQ feedback

- Sensitivities (and input sensitivity) function

$$S = S_u = (I + F_y(s)G(s))^{-1} = (I + G_o(s))^{-1}$$

$$|S(i\omega)| \le 1, \qquad |T(i\omega)| \le 2, \quad \forall\, \omega$$

independently of Q_1 and Q_2

$\Longrightarrow$ good insensitivity to disturbance and model error

Stability margin: LQG (with observer)

Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

Abstract—There are none.

INTRODUCTION

Considerable attention has been given lately to the issue of robustness of linear–quadratic (LQ) regulators. The recent work by Safonov and Athans [1] has extended to the multivariable case the now well-known guarantee of 60° phase and 6 dB gain margin for such controllers. However, for even the single-input, single-output case there has remained the question of whether there exist any guaranteed margins for the full LQG (Kalman filter in the loop) regulator. By counterexample, this note answers that question; there are none.

A standard two-state single-input single-output LQG control problem is posed for which the resulting closed-loop regulator has arbitrarily small gain margin.

EXAMPLE

Consider the following:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u+\begin{bmatrix}1\\ 1\end{bmatrix}w$$

$$y=[1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+v$$

where (x_1,x_2), u, and y denote the usual states, control input, and measured output, and where w and v are Gaussian white noises with intensities $\sigma>0$ and 1, respectively.

Let performance integral have weights

***Abstract*—There are none.**

IEEE Transactions on Automatic Control, Vol 23, No. 4, August 1978.

For LQG use modification: Loop Transfer Recovery (LTR)

Choose $K = \rho B$ ($\rho =$ scalar). Then for the loop gain it holds

$$G_{o,LQG}(s) = L(sI - A + BL + \rho BC)^{-1} \rho BC(sI - A)^{-1} B$$

$$G_{o,LQG}(s) \overset{\rho \to \infty}{\longrightarrow} L(sI - A)^{-1} B = G_{o,LQ}(s)$$

- Instead of "manually" choosing $K = \rho B$ one can, through a certain choice of N and R_1, get a Kalman filter to give $K \approx \rho B$.
- Consequence: one can modify the Kalman filter so that the sensitivity at the input approaches the one you have for full state LQ.
- This however happens at the cost of the filter properties.

Loop Transfer Recovery procedure

Procedure:

1. Choose Q_1 and Q_2 in full state LQ feedback so that the ideal loop gain gives good sensitivity and robustness properties.
2. To compute K in Kalman filter: choose $N = B$ and $R_1 = \alpha R_2$ increasing α until the actual loop gain matches sufficiently close the ideal one.

Procedure requires that the number of inputs and outputs is equal.

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