TSRT09 – Control Theory

Lecture 5: Controller structure and control design

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Summary of lecture 4

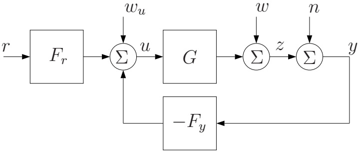
Kalman filter

- Optimal observer
- Requires stochastic models of the disturbances
- Requires solving an algebraic Riccati equation (ARE)

$$K = PC^T R_2^{-1}$$
$$AP + PA^T - PC^T R_2^{-1} CP + NR_1 N^T = 0$$

Summary of lecture 4 (cont'd)

Important transfer functions:



- Closed loop:
  $$G_c = (I + GF_y)^{-1} GF_r$$
  - reference → controlled signal
- Sensitivity function: $S = (I + GF_y)^{-1}$
  - output disturbance → controlled signal
  - model error → controlled signal $\Delta_z \sim S\Delta_G$
- Complementary sensitivity function: $T = (I + GF_y)^{-1} GF_y$
  - measurement error → controlled signal
  - model error → stability $|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|} \ \forall\, \omega$
- Input sensitivity function: $S_u = (I + F_y G)^{-1}$
  - input disturbance → input

Specifications for control design

Task: Choose u so that z follows r as close as possible, in spite of the presence of disturbances w, n, and of uncertainty in the system, while at the same time using reasonable values of u

Translate this into transfer functions:

- error $e = r - z$ small

$$e = (I - G_c)r - Sw + Tn$$

- input not too large

$$u = G_{ru}r + G_{wu}(w + n)$$

- effect of uncertainty small

$$\Delta_z \sim S\Delta_G$$

- closed-loop stability with uncertainty

$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|} \quad \forall\, \omega$$

Specifications for control design

Conditions: design F_y, F_r s.t.

- $I - G_c$ small $\Rightarrow$ controlled variable z follows reference signal r
- S small $\implies$ system disturbance and model errors have small impact on the controlled variable
- T small $\implies$ measurement disturbance has small impact on the controlled variable, and model errors do not compromise stability
- G_{ru} and G_{wu} small $\implies$ input u stays moderate

But observe that

$$S + T = I$$

$$G_c = GG_{ru}$$

$\implies$ conflicts on the specifications!

Lecture 5

Controller structure and control design

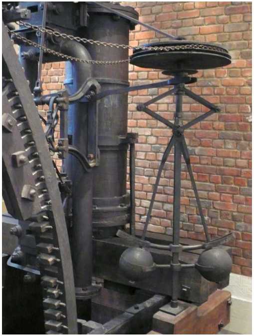
- MIMO systems: who should control who?
  - RGA (Relative Gain Array)
- From structure to design:
  - Decentralized control
  - Decoupled Control
  - IMC (Internal Model Control)
- Synthesis method 1: Linear Quadratic synthesis

In the book: Ch. 8 and 9

Control design

Most successful controller ever: PID

- Timeline:
  - Boulton and Watt 1788: speed control of steam engines, mechanical implementation
  - Hydraulic and pneumatic implementation: late 1800.
  - Electronic implementation: 1930.
  - Computer implemention: 1950.
  - "PID-on-a-chip": 1990s



source: Wikipedia – Andy Dingley

- Modeling
  - First systematic approach (poles): Maxwell 1868.
  - Robust shaping of system gain: Aström och Hägglund 2006.

- Applications: all.

PID and beyond

PIDs:

- It pairs an output and an input
- Can be developed with intuition and experimentation. Result are normally sufficient (good in very simple cases)
- Interpretation in a Bode diagram: lead and lag
- Systematic analysis (poles, zeros, S, T,...) can give control design with high performances
- MIMO systems $\implies$ inputs/outputs must be paired two by two

When is PID not enough:

- most MIMO systems (e.g. when no natural pairing exists)
- in some advanced applications, e.g. when you are given state-based costs to minimize, e.g. linear quadratic design
- nonlinear systems

MIMO systems

Approach: explore structure

- if natural pairings exist $\implies$ decentralized control (each loop is controlled independently)
- if natural pairings do not exist:
  - RGA (Relative Gain Array) – a way to measure interaction
  - Decoupled control: a way to reduce coupling

More advanced control methods

- IMC (Internal Model Control).
- Minimization of quadratic criterion: LQ, LQG.
- Systematic shaping of transfer functions: $\mathcal{H}_2$, $\mathcal{H}_\infty$.
- Nonlinear methods.

MIMO systems: structure-based control design

- $G(s)$ is a $p \times m$ matrix:

$$G(s) = \Big[G_{ij}(s) \Big]_{\substack{i=1,\dots p \\ j=1,\dots,m}}$$

- Control design

$$u = F_r(s)r - F_y(s)y$$

where $F_r(s)$ and $F_y(s)$ are $m \times p$ matrices

$$F_y(s) = \Big[F_{y,ij}(s) \Big]_{\substack{i=1,\dots m \\ j=1,\dots,p}}$$

- Q: who controls who?
- Q: Which y_k use to design u_j?
- Q: How to design F_r and F_y?

MIMO systems: problems

1. Who should control who?
   - G square: $p = m$
   - 'Tall" system: $p > m$

     $$G = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{bmatrix}$$

     Less inputs than outputs: Not all outputs can be controlled perfectly – must prioritize.
   - "Fat" system: $p < m$

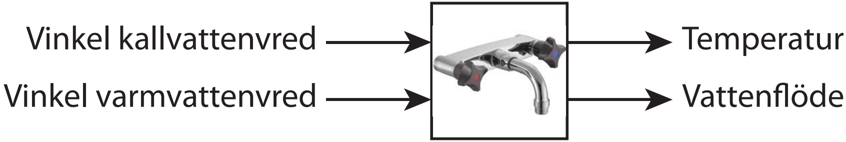
     $$G = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

     More inputs than outputs. How should the control effort be distributed among the inputs?

2. Interactions / cross couplings

Interaction / Cross coupling

- Two-handle water mixer, a system with “hard” crosscoupling.
  - Multiple inputs influence (heavily) each output.
  - Multiple outputs are influenced (heavily) by any input.



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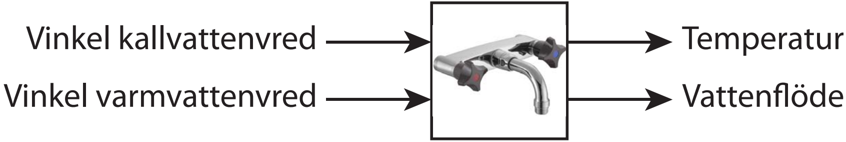

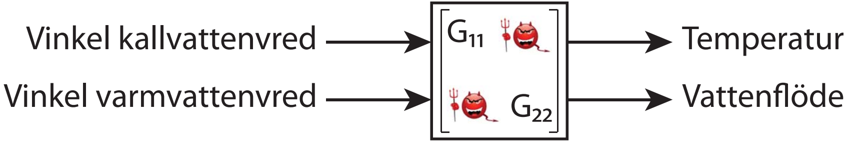

Interaction / Cross coupling (cont'd)

- Single-handle water mixer, a system with "gentle" crosscoupling.
  - Every input influences (almost) just one output.
  - Every output is influenced (almost) by only one input.

Interaction / Cross coupling (cont'd)

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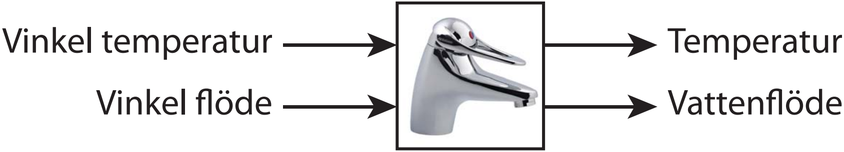
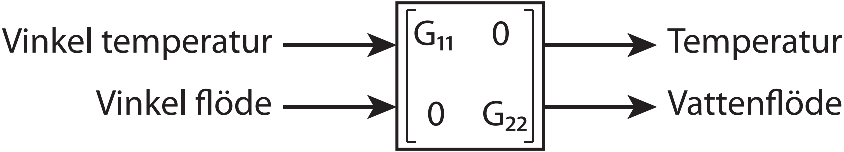




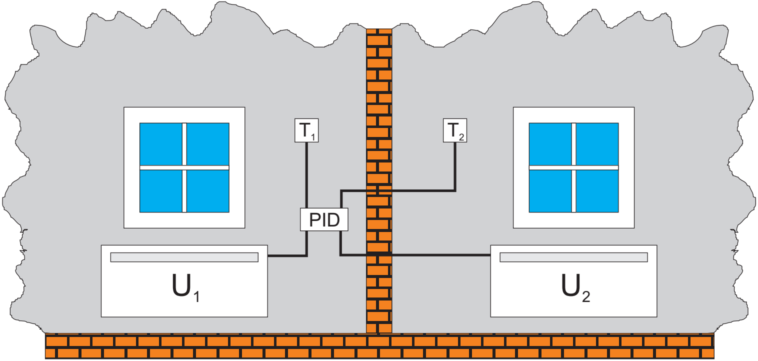
Decentralised regulation

- Build a regulator for a MIMO system by letting *one* output be used by *one* input.
- Result is a number of scalar loops

$$u_j = F_{r,jk}\, r_k - F_{y,jk}\, y_k$$

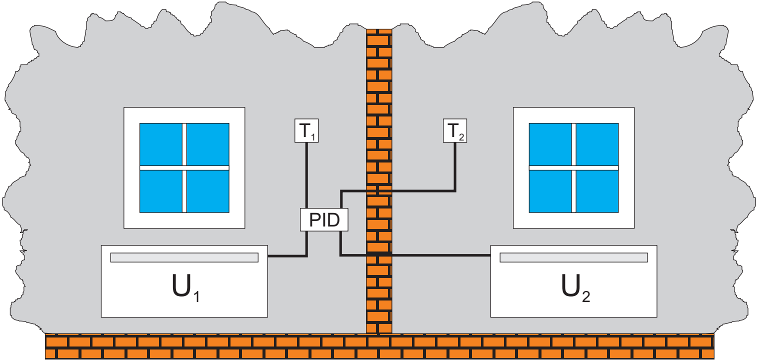
  where each regulator is not "aware" of the presence of the others.
- If G square transfer function matrix $\implies$ can be done
  If G rectangular: disregard some of the signals (?)
- Works when there are natural pairings and cross-couplings are small.
- "Pairing problem": one would like to pair the input and outputs with strongest coupling.
- How do you determine which coupling exist between inputs and outputs?

Example: Temperature regulator



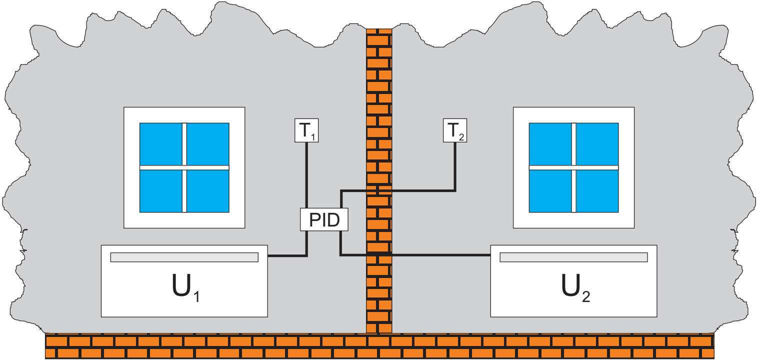
- Two rooms with a wall separating them.
- Temperatures T_1 and T_2 are states and measured variables.
- Each room can warm or cool via U_1 and U_2.

Example: Temperature regulator



$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} \frac{0.005s+0.0002438}{s^2+0.0975s+0.002025} & \frac{0.00009}{s^2+0.0975s+0.002025} \\ \frac{0.00009}{s^2+0.0975s+0.002025} & \frac{0.005s+0.0002438}{s^2+0.0975s+0.002025} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

Example: Temperature regulator



Which sensor should be used by which heat/cold source?

Example: Temperature regulator (cont'd)

Regulator. First pairing:

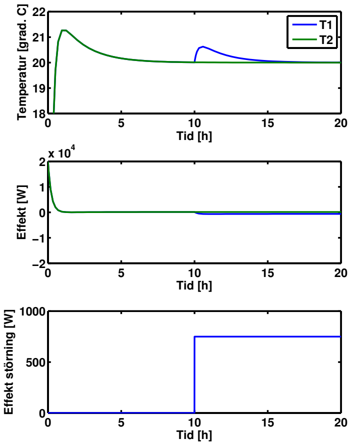
- T_1 is used by U_1.
- T_2 is used by U_2.

Decentralized PI-regulator

$$F_y(s) = \begin{bmatrix} 1000 + \frac{500}{s} & 0 \\ 0 & 1000 + \frac{500}{s} \end{bmatrix}$$

After 10 hours, 10 people enter into room 1.

Stable



Example: Temperature regulator (cont'd)

Regulator. Second pairing:

- T_2 is used by U_1.
- T_1 is used by U_2.

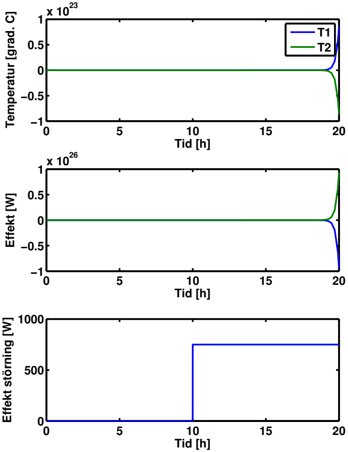
Decentralized PI-regulator

$$F_y(s) = \begin{bmatrix} 0 & 1000 + \frac{500}{s} \\ 1000 + \frac{500}{s} & 0 \end{bmatrix}$$

After 10 hours, 10 people enter into room 1.

Unstable!

Could we have anticipated the problem "analytically"?



RGA (Relative Gain Array)

Consider the following extreme cases

1. output k is controlled from input j and no other input is active

2. output k is controlled from input j and all other outputs are perfectly regulated (e.g. forced to 0).

RGA (Relative Gain Array)

- Construct the ratios between the two gains in the two cases (for all input/output pairs)

- Matematically: elementwise multiplication of G and G^{-T} (or $(G^{\dagger})^T$ if G rectangular)

$$\text{RGA}(G) = G. * (G^{\dagger})^T$$

Decentralized control with RGA

- Properties of RGA
  - function of ω
  - row sum $= 1$
  - column sum $=1$

- Estimating coupling
  - Ideal case (after permutation of inputs/outputs):

$$\text{RGA}(G(i\omega)) = I \ \ \forall\, \omega$$

  - Deviation of RGA$(G(i\omega))$ from $I \implies$ coupling

Decentralized control with RGA

- Practical rules: pair y_k and u_j so that
  - diagonal elements in RGA$(G(i\omega_c))$ are near 1
    ($\omega_c =$ closed-loop cross-over freq.)
  - diagonal elements in RGA$(G(0))$ do not become negative
    (can give instability).
- Decentralized control:
  - treat each loop independently

$$u_j = F_{r,jk}\, r_k - F_{y,jk}\, y_k$$

Example: Temperature regulator with RGA

Example: room temperature regulation

First pairing

- T_1 is used by U_1
- T_2 is used by U_2

$$\text{RGA}(G(i5)) \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{RGA}(G(0)) = \begin{bmatrix} 1.17 & -0.17 \\ -0.17 & 1.17 \end{bmatrix}$$

OK with both rules

Second pairing

- T_2 is used by U_1
- T_1 is used by U_2

$$\text{RGA}(G(i5)) \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{RGA}(G(0)) = \begin{bmatrix} 1.17 & -0.17 \\ -0.17 & 1.17 \end{bmatrix}$$

Violates both rules

Decoupled regulation

- Decentralized control: need "natural pairs" of inputs and outputs
- What if this is not the case? Create them!
- Change of variables for inputs and outputs:

$$\tilde{u} = W_1^{-1} u \qquad\qquad \tilde{y} = W_2 y$$

- Task: design W_1 and W_2 such that the "virtual system":

$$\tilde{G}(s) = W_2(s) G(s) W_1(s)$$

  is as decoupled (i.e., diagonal) as possible

- Then design diagonal regulator $\tilde{F}_y(s)$
- Resulting regulator: $F_y(s) = W_1(s) \tilde{F}_y(s) W_2(s)$

Decoupled regulation (cont'd)

How to choose W_1 and W_2?

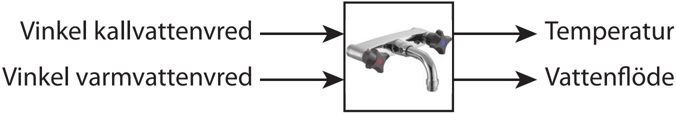
1. "Dynamical" decoupling
   - completely decoupled "virtual system", over all ω
   - s-dependent matrices $W_1(s)$ and $W_2(s)$ are needed
   - often not possible (complicated or non-linear regulator)

2. "Static" decoupling
   - choose *one* frequency at which the system becomes decoupled
     1. steady state $\omega = 0$
     2. cross-over frequency $\omega = \omega_c$

   - Example of steady state decoupling

$$W_1 = G^{-1}(0) \qquad W_2 = I$$

Example: two handle water mixer

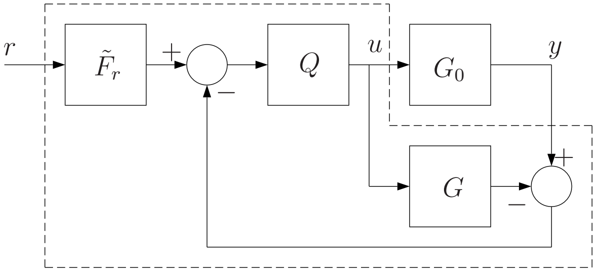


With the right choice of W_1 and W_2 you can make a two-handle water mixer behave as a single-handle mixer. More easily controlled!

$$W_1 = G^{-1}(0) \qquad\qquad W_2 = I$$

Internal Model Control (IMC)

- true system: G_0
- model (assumed stable): G
- "new information": $y - Gu$
- Idea: feed back the "new information" $y - Gu$
- block diagram:



Internal Model Control (IMC)

- IMC: use new information $y - Gu$ as feedback, with a T.F. Q

$$u = -Q \underbrace{(y - Gu)}_{\text{new information}} + Q\tilde{F}_r r$$

or

$$u = -\underbrace{(I - QG)^{-1}Q}_{F_y}\, y + \underbrace{(I - QG)^{-1}Q\tilde{F}_r}_{F_r}\, r$$

- If G stable: all stabilizing controllers $u = -F_y y$ are of the form

$$F_y = (I - QG)^{-1}Q \qquad \forall\, Q \quad \text{stable}$$

IMC design rules: basic idea

- Ideally: choose $Q = G^{-1} \implies G_c = I$

- Complication: unfeasible $F_y \equiv \infty$

- Solution: approximate the inverse appropriately, e.g.

$$Q(s) = \frac{1}{(\lambda s + 1)^n} G^{-1}(s)$$

- Different system properties lead to different suitable approximations of the inverse.

Synthesis of the regulator

Two main approaches:

1. Quadratic weights on the variables + optimization.
   $\Longrightarrow$ "Linear quadratic synthesis": "LQ", "LQG"

2. Direct construction of S, T, in the frequency domain:
   $\Longrightarrow$ "'$\mathcal{H}_2$", '$\mathcal{H}_\infty$"

A warm-up problem

- State space model
- fully deterministic problem

$$
\begin{aligned}
\dot{x} &= Ax + Bu, \qquad x(0) \text{ given} \\
z &= Mx \\
y &= x
\end{aligned}
$$

Minimization of a quadratic criterion

Problem: Full state LQ (Linear Quadratic) optimal control

Find a $u = -Lx$ that solves

$$\min J = \int_0^\infty (z^T Q_1 z + u^T Q_2 u) dt$$

$$\text{s. t. } \dot{x} = Ax + Bu, \qquad x(0) \text{ given}$$

$$z = Mx$$

$$\text{and } A - BL \text{ is stable}$$

Solution: $u = -Lx, \quad L = Q_2^{-1} B^T S,$ where S solves the ARE

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0$$

Solving the algebraic Riccati equation

- Is it always possible to solve

$$A^T S + SA + \bar{Q}_1 - SBQ_2^{-1}B^T S = 0, \qquad \bar{Q}_1 = M^T Q_1 M$$

so that the state matrix of the regulated system

$$A - BQ_2^{-1}B^T S$$

has its eigenvalues in the left half of the complex plane?

Solving the algebraic Riccati equation (cont'd)

- **Example**:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{Q}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_2 = 1$$

```
» [L,S,E]=lqr(A,B,Q1,Q2)
??? Error using ==> lti.lqr
The plant model cannot be stabilized by feedback or
the optimal design problem is ill posed.
```

Solvability of the algebraic Riccati equation

If

1. $Q_1 \geq 0$, $Q_2 > 0$
2. (A, B) is stabilizable (unstable states are controllable)
3. $(A, \bar{Q}_1)$ is detectable (unstable states can be “seen” in the cost)

then there exists a unique solution $S = S^T \geq 0$ of

$$A^T S + SA + \bar{Q}_1 - SBQ_2^{-1}B^T S = 0$$

with $L = Q_2^{-1}B^T S$, that minimizes the criterion

$$J = \int_0^\infty (x^T \bar{Q}_1 x + u^T Q_2 u)dt$$

and

$$A - BQ_2^{-1}B^T S$$

has all eigenvalues strictly in the left half of the complex plane.

Solving the algebraic Riccati equation (cont'd)

- **Example**:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{Q}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_2 = 1$$

- **Example**:

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Similarities between Kalman filter and LQ

Kalman filter:

$$AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP = 0$$
$$K = PC^TR_2^{-1} \quad (\Leftrightarrow K^T = R_2^{-1}CP)$$

$$P \longleftrightarrow S$$
$$A^T \longleftrightarrow A$$
$$C^T \longleftrightarrow B$$
$$N^T \longleftrightarrow M$$
$$R_1 \longleftrightarrow Q_1$$
$$R_2 \longleftrightarrow Q_2$$
$$K^T \longleftrightarrow L$$

LQ-regulator:

$$A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS = 0$$
$$L = Q_2^{-1}B^TS$$

Both problems require solving an ARE

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