TSRT09 – Control Theory

Lecture 4: Kalman filter & the closed-loop system

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Summary of lecture 3

- Covariance matrix (at τ):

$$R_u(\tau) = E[u(t)u(t-\tau)^T] = \int_{-\infty}^{+\infty} u(t)u(t-\tau)^T d\tau$$

- Spectrum:

$$\Phi_u(\omega) = \int_{-\infty}^{\infty} R_u(\tau)e^{-i\omega\tau} \; d\tau$$

- Signal "size" (covariance matrix at 0):

$$R_u = R_u(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_u(\omega) \, d\omega$$

Summary of lecture 3 (cont'd)

- White noise $v(t)$:

$$\Phi_v(\omega) = \text{constant} = R$$
$$R_v(\tau) = R\,\delta(\tau)$$

- Spectra factorization: every spectrum can be though as being generated by a white noise passed through a linear system

$$y = Gv \quad \Rightarrow \quad \Phi_y(\omega) = G(i\omega)\,\underbrace{\Phi_v(\omega)}_{R}\,G^T(-i\omega)$$

Summary of lecture 3 (cont'd)

- White noise in state space form:

$$\dot{x} = Ax + Nv$$

v white noise of intensity/spectrum R.

- Covariance matrix of x: $\Pi_x = R_x(0)$ solution of

$$A\Pi_x + \Pi_x A^T + NRN^T = 0 \qquad \text{Lyapunov equation}$$

Summary of lecture 3 (cont'd)

- System

$$\dot{x} = Ax + Bu + Nv_1,$$
$$y = Cx + Du + v_2$$

v_1, v_2 white noises

- Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x} - Du)$$

- Estimation error

$$\dot{\tilde{x}} = (A - KC)\tilde{x} + Nv_1 - Kv_2$$

Lecture 4

- Kalman filter

In the book: Ch. 5

- PART II: LINEAR CONTROL THEORY
  - Description of the closed-loop system

In the book: Ch. 6

Kalman filter

- System $$\dot{x} = Ax + Bu + Nv_1,$$ $$y = Cx + Du + v_2$$

  v_1, v_2 uncorrelated white noises with intensity R_1, R_2

- Observer: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x} - Du)$

Kalman filter = optimal observer, of gain:

$$K = PC^T R_2^{-1}$$

where P is given by the algebraic Riccati equation (ARE):

$$AP + PA^T - PC^T R_2^{-1} CP + NR_1 N^T = 0$$

- It minimizes covariance of estimation error $P = E[\tilde{x}(t)\tilde{x}^T(t)]$ subject to $A - KC$ stable

When can the algebraic Riccati equation be solved?

If

1. $R_1 \geq 0$, $R_2 > 0$
2. (A, C) detectable
   Meaning:
   - the unstable part of the system is observable
   - observer error (i.e., $A - KC$) can be rendered stable
3. (A, NR_1N^T) stabilizable
   Meaning:
   - the unstable part of the system is "controllable from noise"
   - more mathematical/technical condition

then there is a solution $P = P^T \geq 0$ to the algebraic Riccati equation s. t. all eigenvalues of $A - KC = A - PC^T R_2^{-1} C$ have real part < 0.

Properties of the Kalman filter

1. Kalman filter is the observer that minimizes the covariance matrix P for the estimation error $\tilde{x}$

2. Kalman filter minimizes the mean square error (i.e., the variance of $\tilde{x}$) among all causal filters

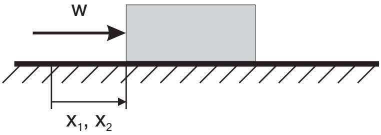
3. Error $\tilde{x}(t)$ is uncorrelated with estimates $\hat{x}(s)$, $s \leq t$

4. "Innovation": $\nu(t) = y(t) - C\hat{x}(t) - Du(t)$ is a white noise (completely unpredictable)

Application of Kalman filter: sensor fusion

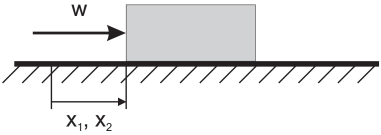
- Often one has to consider the same signal from different sensors having different accuracies.
- In a navigation system for ship, airplane, car, ... one considers simultaneously
  - Position information: GPS, radio beacons, optical signals...
  - Velocity information: doppler radars, tachometers, gyro (angular velocity),...
  - Acceleration measurements: accelerometers,....
  - ....
- An important point in this context is that the weak aspects of a sensor can be compensated by another sensor that does not have the *same* weak aspects.

Simple sensor fusion: position – velocity



- Movement in 1-dimension: x_1 position, x_2 velocity.
- Acceleration varies randomly around 0: modeled as a white noise, w.
- Position and velocity measurement: y_1 resp. y_2.
- Position and velocity are measured with some measurement error: v_1 resp. v_2.

Simple sensor fusion: position – velocity



Model:

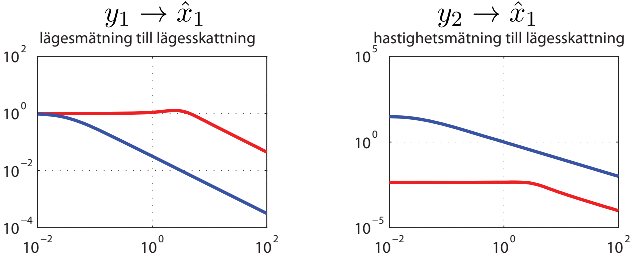
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

$$y = x + v$$

$$R_1 = 1, \quad R_2 = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

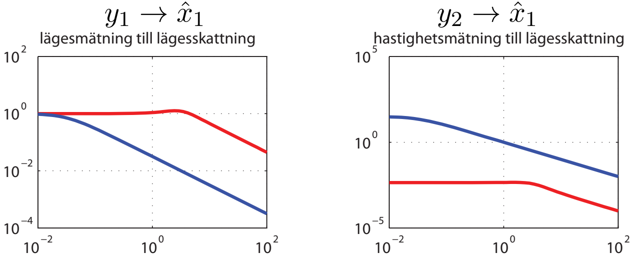
w, v uncorrelated

Kalman filter: Bode diagram - Position estimation



- Good position meas. & bad velocity meas.
  (r_1 small, r_2 big): $\hat{x}_1 \approx y_1 + 0 \cdot y_2$
- Bad position meas. & good velocity meas.
  (r_1 big, r_2 small): $\hat{x}_1 \approx 0 \cdot y_1 + \int y_2$

Kalman filter: Bode diagram - Position estimation



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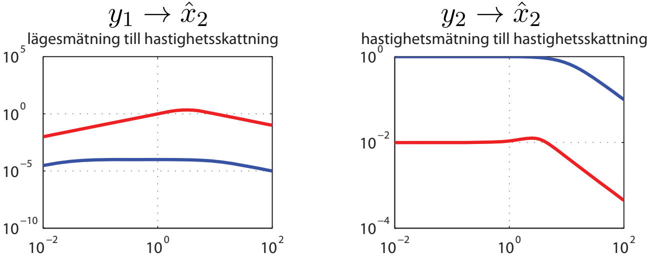
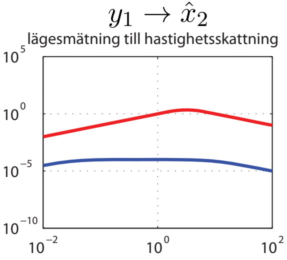
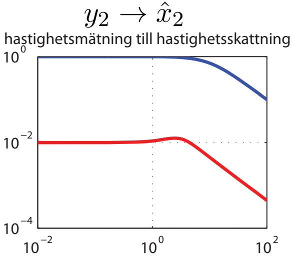
Kalman filter: Bode diagram - Velocity estimation



- Good position meas. & bad velocity meas.
  (r_1 small, r_2 big): $\hat{x}_2 \approx \frac{d}{dt} y_1 + 0 \cdot y_2$
- Bad position meas. & good velocity meas.
  (r_1 big, r_2 small): $\hat{x}_2 \approx 0 \cdot y_1 + y_2$

Kalman filter: Bode diagram - Velocity estimation





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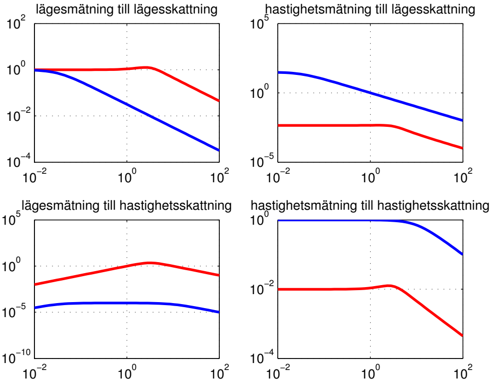
Kalman filter: Bode diagram - Summary



In other words: high noise level in a measurement $\Rightarrow$ Filter does not “trust” too much that measurement

PART II: LINEAR CONTROL THEORY

- Lecture 4: The closed-loop system
- Lecture 5: Controller structure
- Lecture 6: LQ-regulation
- Lecture 7: Loopshaping
- Lecture 12: Specifications and limitations

    - Canonical block diagram
    - Stability for the closed-loop system
    - Sensitivity & Robustness
    - Specifications

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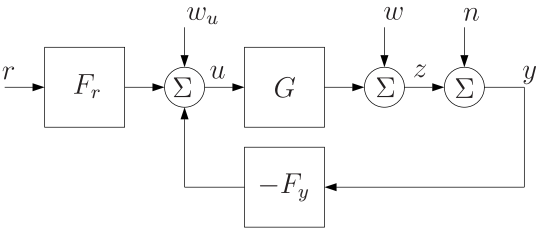
In the book: Ch. 6

The closed-loop system: important signals



Signals

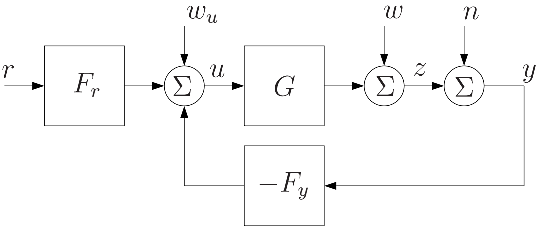
- r = reference signal
- u = control signal
- z = regulated variable
- y = measured signal

Disturbances

- w = system disturbance (at the output)
- n = measurement noise
- w_u = disturbance at the input

- Often it is $y = z + n$
- Control law: $u = F_r(s)r - F_y(s)y$

The closed-loop system: important transfer functions



Relationships among signals:

$$z = G_c r + Sw - Tn + GS_u w_u$$

$$u = S_u F_r r - S_u F_y (w + n) + S_u w_u$$

The closed-loop system: important transfer functions

Transfer functions:

- Closed-loop system ("main T.F.")

$$G_c = (I + GF_y)^{-1} GF_r$$

- Sensitivity function

$$S = (I + GF_y)^{-1}$$

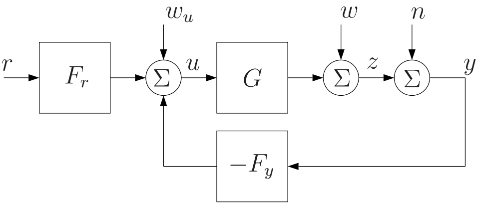
- Complementary sensitivity function

$$T = (I + GF_y)^{-1} GF_y$$

- Input sensitivity function

$$S_u = (I + F_y G)^{-1}$$

Stability for the closed-loop system



- Basic control design: choose F_y so that G_c is stable
- Q: in presence of disturbances, is it enough to have G_c stable?
- Input-output stability: stability from all inputs (r, w, n, w_u) to all signals (u, z, y)
- Q: when is the closed loop system stable from all inputs to all outputs?

Stability for the closed-loop system

- Consider the closed-loop system as a system with inputs w_u, w and output u, y (for $r = 0$, $n = 0$).

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} SG & S \\ S_u & -S_u F_y \end{bmatrix} \begin{bmatrix} w_u \\ w \end{bmatrix} \tag{1}$$

- "Internal stability": if the 4 transfer functions

$$SG, \;\; S, \;\; S_u, \;\; S_u F_y$$

are all stable (and F_r stable) then the entire system is stable

Sensitivity (to model errors)

- In practice the system is not known exactly.
- nominal system: G
- true system: $G_0 = (I + \Delta_G)G \qquad \Delta_G =$ model error
- true output $z_0 = (I + \Delta_z)z$
- relative output error

$$\Delta_z = S_0 \Delta_G, \qquad S_0 = (I + G_0 F_y)^{-1}$$

- In practice: since G_0 is not known, S_0 must be approximated by

$$S_0 \simeq S = (I + G F_y)^{-1}$$

- Interpretation: S is the gain from model error to rel. output error
  S should be small when the model error Δ_G is big

Robustness (to model errors)

- How big a model error Δ_G can be tolerated without compromising the stability of the closed-loop system?

- If
$$\|\Delta_G T\|_\infty < 1$$
then the closed-loop system is still stable.

- In its turn, this is true if
$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}, \quad \text{all } \omega$$

- Interpretation: T should be small when the model error Δ_G is big

Specifications for control design

Task: Choose u so that z follows r as close as possible, in spite of the presence of disturbances w, n, and of uncertainty in the system, while at the same time using reasonable values of u

Translate this into transfer functions:

- error $e = r - z$ small

$$e = (I - G_c)r - Sw + Tn$$

- input not too large

$$u = G_{ru}r + G_{wu}(w + n)$$

- effect of uncertainty

$$\Delta_z = S_0 \Delta_G \sim S\Delta_G$$

- closed-loop stability with model error

$$\|\Delta_G T\|_\infty < 1$$

Specifications for control design

Conditions:

- $I - G_c$ small $\Rightarrow$ controlled variable z follows reference signal r
- S small $\implies$ system disturbance and model errors have small impact on the controlled variable
- T small $\implies$ measurement disturbance has small impact on the controlled variable, and model errors do not compromise stability
- G_{ru} and G_{wu} small $\implies$ input u stays moderate

But observe that

$$S + T = I$$

$$G_c = GG_{ru}$$

$\implies$ conflicts on the specifications!

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Lecture 4

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