

# TSRT09 – Control Theory

Lecture 3: Disturbance models

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## Summary of lecture 2

- Most of the theory for SISO systems generalizes to MIMO
- Significant differences exist in
  - minimal realization
  - controllability and observability canonical forms
  - computation of poles and zeros in transfer function matrices
- Poles:
  1. Eigenvalues of a state matrix (for a minimal realization)
  2. Roots of the *pole polynomial* = least common multiple of all denominators of all minors (i.e., determinant of square submatrices) of  $G(s)$ .
- Zeros:
  1. Roots of the *zero polynomial* = greatest common divisor for the numerators of the maximal minors of  $G(s)$
  2. State space criterion:

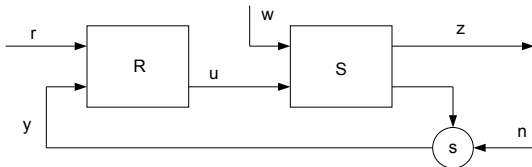
the matrix  $M(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}$  loses rank for  $s = \text{zero}$

## Lecture 3

- Description of disturbances
- Standard form of system with disturbances in state space form
- Observer
- Kalman filter

In the book: Ch. 5

# What is a disturbance?



## Two main types of disturbances

### 1. System disturbance: $w$

- affect the system to be controlled
- variations in the process
- nonmodeled dynamics
- extra non-controllable inputs

### 2. Measurement disturbance / noise: $n$

- affects  $y$  but not  $z$
- measurement noise in the sensors
- more general sensors disturbances

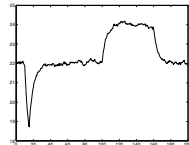
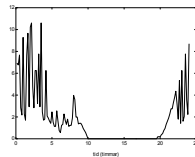
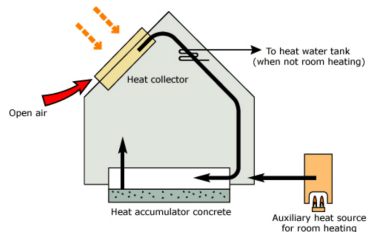
# Example: solar heated house

## System disturbances:

- solar irradiation
- external temperature, wind
- are the windows open?
- how many people in the house?

## Measurement disturbance / error

- sensor round-off
- gradient of temperature across different rooms



# Models for disturbances

Adding disturbances to our models:

- in transfer functions:

$$Y(s) = G(s)(U(s) + W(s)) + N(s)$$

- in state space models

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Nw(t) \\ y(t) &= Cx(t) + Du(t) + n(t)\end{aligned}$$

## Models for the disturbances

- Time domain: covariance
- Frequency domain: spectrum

## Signal size measure

- Our signal size measure:

$$\|z\|_2^2 = \int_{-\infty}^{\infty} |z(t)|^2 dt \quad |z|^2 = z^T z \quad \text{“energy”}$$

- If the integral does not converge, one can use the measure

$$\|z\|_e^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N |z(t)|^2 dt \quad \text{“power”}$$

This is a rough description (a scalar)

## More informative measure for vectors of signals

- How do the components of  $z$  relate to each other? Matrix measure

$$R_z = \int_{-\infty}^{+\infty} z(t)z^T(t) dt \quad \text{"energy"}$$

$$R_z = \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N z(t)z^T(t) dt \quad \text{"power"}$$

- More general: "covariance (at  $\tau$ )"

$$R_z(\tau) = \int_{-\infty}^{+\infty} z(t)z^T(t - \tau) dt$$

- $\implies R_z = \text{"covariance (at 0)"}'$

$$R_z = R_z(0)$$



## Underlying idea: stationary stochastic processes

- at each  $t$ :  $z(t)$  = vector of random var.  $\implies$  **stochastic process**
- stationary stochastic process: description does not depend on  $t$
- to describe a stochastic process:

1. mean

$$m_z = E[z(t)]$$

“mathematical expectation”

2. variance

$$\text{Var}[z(t)] = E[(z(t) - m_z)^2]$$

“deviation from the mean”

3. covariance

$$R_z(\tau) = E[(z(t) - m_z)(z(t - \tau) - m_z)^T]$$

matrix, “signal compared with itself, but shifted”

- for us: zero mean  $m_z = 0$

# Autocovariance vs. cross-covariance

## Example

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \implies R_z(\tau) = \begin{bmatrix} R_{z_1, z_1}(\tau) & R_{z_1, z_2}(\tau) \\ R_{z_2, z_1}(\tau) & R_{z_2, z_2}(\tau) \end{bmatrix}$$

- diagonal

$$R_{z_i, z_i}(\tau) = \int_{-\infty}^{+\infty} z_i(t) z_i(t - \tau) dt \quad \text{"autocovariance"}$$

- off-diagonal

$$R_{z_i, z_j}(\tau) = \int_{-\infty}^{+\infty} z_i(t) z_j(t - \tau) dt \quad \text{cross-covariance}$$

$$R_{z_i, z_j} = 0 \implies z_i \text{ and } z_j \text{ are uncorrelated}$$

Spectrum = "size distribution" at different frequencies

Spectrum of  $z(t)$

$\Phi_z(\omega)$  = "Energy (or power) content of  $z$  at frequency  $\omega$ "

- $\Phi_z(\omega)$  = matrix function
- if  $Z(i\omega)$  = Fourier transform of  $z(t)$

$$\Phi_z(\omega) = Z(i\omega)Z^*(i\omega)$$

- using Parseval

$$R_z = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_z(\omega) d\omega$$

# Spectrum

- Alternatively:  $\Phi_z(\omega)$  Fourier transform of  $R_z(\tau)$

$$\Phi_z(\omega) = \int_{-\infty}^{\infty} R_z(\tau) e^{-i\omega\tau} d\tau$$

- $R_z(\tau)$  and  $\Phi_z(\omega)$  form a Fourier transform pair

$$\Phi_z(\omega) = \mathcal{F}[R_z(\tau)], \quad R_z(\tau) = \mathcal{F}^{-1}[\Phi_z(\omega)]$$

- when computed in  $\tau = 0$ :

$$R_z = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_z(\omega) d\omega$$

# Cross spectrum

## Example

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \implies \Phi_z(\omega) = \begin{bmatrix} \Phi_{z_1, z_1}(\omega) & \Phi_{z_1, z_2}(\omega) \\ \Phi_{z_2, z_1}(\omega) & \Phi_{z_2, z_2}(\omega) \end{bmatrix}$$

- Diagonal element  $\Phi_{z_i, z_i}(\omega)$  measures the energy content of  $z_i$  at frequency  $\omega$
- Off-diagonal element  $\Phi_{z_i, z_j}(\omega)$  measures the coupling between  $z_i$  and  $z_j$  at frequency  $\omega$ 
  - $\Phi_{z_i, z_j}(\omega) =$  **cross spectrum** between  $z_i$  and  $z_j$

$$\Phi_{z_i, z_j}(\omega) = \int_{-\infty}^{\infty} R_{z_i, z_j}(\tau) e^{-i\omega\tau} d\tau$$

- $\Phi_{z_i, z_j}(\omega) \equiv 0 \implies z_i$  and  $z_j$  are **uncorrelated** ( $\iff R_{z_i, z_j} = 0$ )

White noise = unpredictable signal

White noise = signal with constant spectrum

$$\Phi_z(\omega) = R = \text{const} \quad \forall \omega$$

- Covariance:

$$R_z(\tau) = \delta(\tau)R = \begin{cases} R & \text{if } \tau = 0 \\ 0 & \text{if } \tau \neq 0 \end{cases}$$

- Meaning: old values give no information on the future values
- $\implies$  white noise is **unpredictable**.

# Spectra and transfer functions



If  $u$  has spectrum  $\Phi_u(\omega)$ , what is  $\Phi_y(\omega) =$  spectrum of  $y$ ?

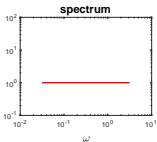
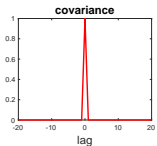
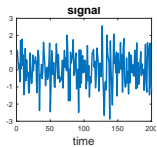
$$Y(i\omega) = G(i\omega)U(i\omega), \quad U, Y \text{ Fourier transforms}$$

$$\Phi_y(\omega) = Y(i\omega)Y(i\omega)^* = G(i\omega)U(i\omega)U(i\omega)^*G(i\omega)^*$$

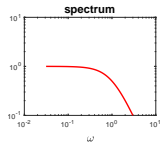
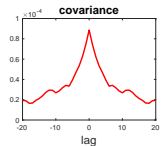
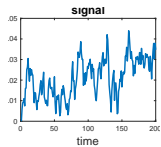
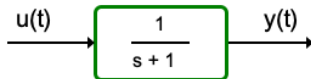
or

$$\Phi_y(\omega) = G(i\omega)\Phi_u(\omega)G(i\omega)^*$$

# Example of spectrum

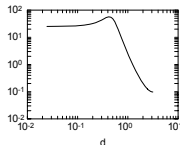
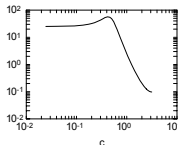
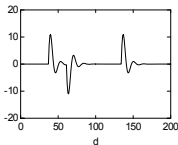
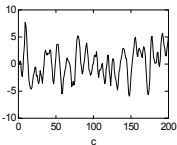
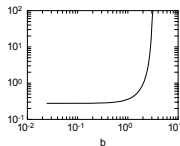
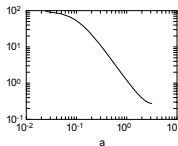
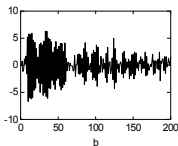
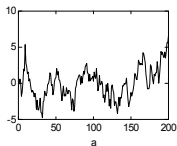


$$Y(s) = \frac{1}{s+1}U(s)$$





# A few “colored noises” and their spectra



$$u \in \mathcal{N}(0, \lambda) \quad u = \begin{cases} 0 & \text{with prob. } 0.8 \\ \in \mathcal{N}(0, \lambda) & \text{with prob. } 0.2 \end{cases}$$

# Disturbance from white noise: spectral factorization



"Reverse problem": given  $y(t)$  of spectrum  $\Phi_y(\omega)$ , choose  $G(s)$  and a white noise  $u(t)$  of spectrum  $\Phi_u(\omega) = R$  so that

$$Y(s) = G(s)U(s) \quad \text{and} \quad \Phi_y(\omega) = G(i\omega)RG(i\omega)^*$$

$\implies$  spectral factorization

- Scalar case. If  $u$  and  $y$  are scalar and  $\Phi_y$  is a rational function: easy to solve with  $G$  stable.
- Matrix case. If  $u$  and/or  $y$  are matrices: more difficult

# State space model with disturbances

- state space model with disturbances

$$\dot{x}(t) = Ax(t) + Bu(t) + Nw(t)$$

$$y(t) = Cx(t) + Du(t) + n(t)$$

- $w(t)$  and  $n(t)$  are in general not white  $\implies$  “colored noise”
- Task: model should contain all information useful to predict future values  $\implies$  also “colored noise” contributes
- “Whitening” the disturbances:  $w$  and  $n$  are outputs of the linear systems  $w = G_w v_1$ ,  $n = G_n v_2$ , where  $v_1$ ,  $v_2$  are white

## State space model with disturbances

- Expressing  $G_w, G_n$  in state space form

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) + \bar{N}v_1(t)$$

$$y(t) = \bar{C}\bar{x}(t) + Du(t) + v_2(t)$$

$\implies \bar{x}$  is extended to include the states in  $G_w, G_n$

$\implies$  “new” disturbance  $v_1$  and measurement noise  $v_2$  are white noises  $\implies$  unpredictable

## “Size” of $x$ induced by white noise

$$\dot{x} = Ax + Nv_1, \quad \Phi_{v_1}(\omega) = R$$

- Transfer function from  $v_1$  to  $x$  is  $(sI - A)^{-1}N$ .
- Spectrum is then

$$\Phi_x(\omega) = (i\omega I - A)^{-1}NRN^T(-i\omega I - A)^{-T}$$

- “Size” of  $x$  (i.e., covariance matrix of  $x$ )

$$\Pi_x = R_x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) d\omega$$

$\Pi_x$  can be computed by solving the **Lyapunov equation**

$$A\Pi_x + \Pi_x A^T + NRN^T = 0$$

## State estimation: observer

$$\dot{x}(t) = Ax(t) + Bu(t) + Nv_1(t)$$

$$y(t) = Cx(t) + Du(t) + v_2(t)$$

- **Task:** estimate  $x$  from the measurement  $y$
- $\implies$  **state observer**  $\hat{x}$
- Dynamics of  $\hat{x}$ :

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t) - Du(t))$$

- Estimation error:  $\tilde{x}(t) = x(t) - \hat{x}(t)$
- Dynamics of  $\tilde{x}$ :

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t) + Nv_1(t) - Kv_2(t)$$

- Choice of  $K$  decides how fast the estimation error converges and also how much the estimation is influenced by the measurement error.  $\implies$  Trade off

# Kalman filter: the optimal observer

$$K = PC^T R_2^{-1}$$

where  $P$  is determined from

$$AP + PA^T - PC^T R_2^{-1} CP + NR_1 N^T = 0 \quad \text{Riccati equation}$$

- Optimality 1: The linear filter that has the least mean square error (for all noise for which the covariance is defined)
- Optimality 2: (for Gaussian noise) The best (linear or nonlinear) filter for many criteria.  $\hat{x}$  is the conditional expectation of  $x$ , given the observations.

Disturbance: something "kicking the system"





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Lecture 3

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