# TSRT09 – Control Theory

## Lecture 2: Description of linear systems

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# Summary of Lecture 1

Signal norm:

$$\|z\|_2^2 = \int_{-\infty}^{\infty} z^*(t)z(t)\ dt$$

Gain of the system $\mathcal{S}$ :

$$\|\mathcal{S}\| = \sup_u \frac{\|y\|_2}{\|u\|_2}$$

# Summary of Lecture 1 (cont'd)

Multivariable linear system with transfer function matrix $G(s)$:

- $\underline{\sigma}(G(i\omega)) \leq \frac{|Y(i\omega)|}{|U(i\omega)|} \leq \bar{\sigma}(G(i\omega)) = |G(i\omega)|$

- $|G(i\omega)|$= largest singular value of $G(i\omega)$

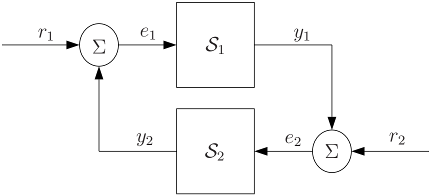
- $\|G\|_\infty$ = largest singular value of $G(i\omega)$ for all $\omega$
  $= \mathcal{H}_\infty$ norm of $G$
  $= \sup_\omega |G(i\omega)|$

- $\frac{\|y\|_2}{\|u\|_2} \leq \|G\|_\infty$

**Prerequisite:** $G$ has all poles strictly in the left half plane.

# Left from Lecture 1: small gain theorem

$\mathcal{S}$ is input-output stable
if $\|\mathcal{S}\|$ is finite




**Theorem:** (Small gain theorem) Two stable systems $\mathcal{S}_1$ and $\mathcal{S}_2$ feedback coupled as in the figure give a closed loop system which is input-output stable if

$$||\mathcal{S}_2|| \cdot ||\mathcal{S}_1|| < 1$$

**Corollary:** For linear systems, the criterion simplifies to

$$||\mathcal{S}_2\mathcal{S}_1|| < 1$$

# PART I: LINEAR SYSTEMS

- Lecture 2: MIMO systems
    1. Representation
    2. Properties
- Lecture 3: Disturbances
- Lecture 4: Kalman filter

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- Lecture 3: Disturbances
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In the book: Ch. 2 and 3

# Representation of MIMO linear systems

**New:** $u$ and $y$ are vectors (scalars in the basic control course)


MIMO system representations:

- State space form
- Transfer function (scalar $\rightarrow$ matrix)
- Impulse response – weighting function (scalar $\rightarrow$ matrix)

# State space form for MIMO systems

- State space model

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$B$ has multiple columns. $C$ has multiple rows. $D$ is a matrix.

- Relationship with transfer function matrix

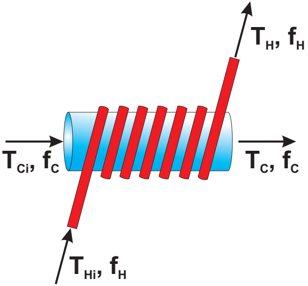
$$G(s) = C(sI - A)^{-1}B + D$$

- Relationship with impulse response

$$g(t) = \mathcal{L}^{-1}\left[G(s)\right]$$

*Completely analogous to the basic control course*

# An example: heat exchanger




Variables:
$T_{H_i}, T_H =$ hot temp.
$T_{C_i}, T_C =$ cold temp.

Parameters:
$V_H, V_C =$ volumes
$f_H, f_C =$ flows
$\beta =$ heat exchange coeff.

$$V_C \frac{dT_C}{dt} = f_C(T_{C_i} - T_C) + \beta(T_H - T_C)$$
$$V_H \frac{dT_H}{dt} = f_H(T_{H_i} - T_H) - \beta(T_H - T_C)$$

# Heat exchanger (cont'd)

- If $x = \begin{bmatrix} T_C \\ T_H \end{bmatrix}$ and $u = \begin{bmatrix} T_{C_i} \\ T_{H_i} \end{bmatrix}$

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -(f_C + \beta)/V_C & \beta/V_C \\ \beta/V_H & -(f_H + \beta)/V_H \end{bmatrix} x + \begin{bmatrix} f_C/V_C & 0 \\ 0 & f_H/V_H \end{bmatrix} u \\ y &= x \end{aligned}$$

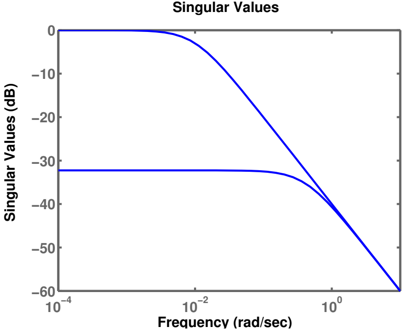
- Use the numerical values $f_C = f_H = 0.01$ ($m^3$/min), $\beta = 0.2$ and $V_H = V_C = 1$ ($m^3$), which gives

$$\begin{aligned} \dot{x} = Ax + Bu &= \begin{bmatrix} -0.21 & 0.2 \\ 0.2 & -0.21 \end{bmatrix} x + \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} u \\ y = Cx &= x \end{aligned}$$

# Heat exchanger (cont'd)

$$G(s) = C(sI - A)^{-1}B$$
$$= \frac{0.01}{(s+0.01)(s+0.41)} \begin{bmatrix} s+0.21 & 0.2 \\ 0.2 & s+0.21 \end{bmatrix}$$

Singular values:



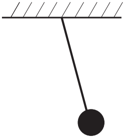
# Properties of MIMO linear systems

MIMO system properties:

- Stability
- Controllability and Observability
- Change of variables and canonical forms
- Poles and zeros

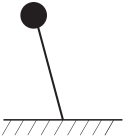
# Examples of stable/unstable system

- "Normal" pendulum without friction
  $\Longrightarrow$ (marginal) stability



- "Normal" pendulum with friction
  $\Longrightarrow$ asymptotic stability

- Inverted pendulum
  $\Longrightarrow$ unstable



# Stability

The equilibrium point $x^* = 0$ of the system $\dot{x} = f(x)$ is

- (marginally) stable if for each $\epsilon > 0$ there exists a $\delta > 0$ such that
$$|x(0)| \leq \delta \implies |x(t)| \leq \epsilon \quad t \geq 0$$
- asymptotically stable if in addition $\lim_{t\to\infty} x(t) = 0$, for all $x(0)$
- unstable if it is not stable

*Exactly as in the basic control course*

# Stability and eigenvalues

Consider a linear system $\dot{x} = Ax$

**Eigenvalue** conditions for stability:

- Asymp. stability $\iff$ all eigenvalues of $A$, $\lambda_j$, have $\text{Re}(\lambda_j) < 0$
- If $A$ has all eigenvalues $\lambda_j$ with $\text{Re}(\lambda_j) \leq 0$ and all eigenvalues on the imaginary axis are simple $\implies$ (marginal) stability
- If $A$ has some eigenvalue $\lambda_j$ with $\text{Re}(\lambda_j) > 0 \Rightarrow$ instability

*Exactly as in the basic control course*

# BIBO-stability

System

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is BIBO-stable if bounded $u$ gives bounded $y$ when $x(0) = 0$.

- Input-output stability $\iff$ BIBO stability
- All eigenvalues of $A$, $\lambda_j$, have $\text{Re}(\lambda_j) < 0 \implies$ BIBO stability

*Completely analogous to the basic control course*

# Controllability

- $\bar{x}$ is *controllable* if there exists $u(t)$ such that $x = 0$ is driven to $x = \bar{x}$.
- System is controllable if all states are controllable.
- **Controllability matrix**

$$S = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}, \quad (n = \text{dim } x)$$

- Controllability $\iff$ $S$ has rank $n$
- **Controllable subspace** = range space of $S$: the vector space spanned by the columns of $S$. "The states we can reach with the input $u(t)$".

*Completely analogous to the basic control course*

# Controllability and the eigenvalues of $A - BL$

- If the system is controllable the eigenvalues of $A - BL$ can be placed arbitrarily
- Why are the eigenvalues of $A - BL$ of interest?
- If the system

  $$\dot{x} = Ax + Bu$$

  is coupled with the state feedback

  $$u = -Lx$$

  the closed loop system becomes

  $$\dot{x} = (A - BL)x$$

  $\implies A - BL$ becomes the "A-matrix" of the closed-loop system

*Completely analogous to the basic control course*

# Observability

- $\bar{x}$ is *non-observable* if the output $y$ is identically zero even though $x(0) = \bar{x}$, (and $u(t) \equiv 0$).
- The system is *observable* if it does not have any non-observable state.
- **Observability matrix**

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}, \quad (n = \text{dim } x)$$

- Observability $\iff$ $\mathcal{O}$ has rank $n$
- **Non-observable subspace** = null space of $\mathcal{O}$. "The states that cannot be seen from $y$"

*Completely analogous to the basic control course*

# Observability and the eigenvalues of $A - KC$

- If the system is observable then the eigenvalues of $A - KC$ can be placed arbitrarily
- Why are the eigenvalues of $A - KC$ of interest?
- If the state of the system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is computed with the help of an observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

the equation for the estimation error $\tilde{x} = x - \hat{x}$ becomes

$$\dot{\tilde{x}} = (A - KC)\tilde{x}$$

$\Longrightarrow A - KC$ becomes the "A-matrix" of the estimation error

*Completely analogous to the basic control course*

# Controllability and observability: examples

- **Example 1** : heat exchanger

$$\dot{x} = \begin{bmatrix} -0.21 & 0.2 \\ 0.2 & -0.21 \end{bmatrix} x + \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} u$$

$$y = x$$

$\Longrightarrow$ Controllable and observable

- **Example 2**:

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$\Longrightarrow$ Controllable but not observable

# Minimal realization

- Only the controllable and observable part of the system can be seen in $G(s)$
- A realization $(A,\, B,\, C,\, D)$ is minimal if it is both controllable and observable
- Minimal realization: the state dimension $n$ is the least possible that realizes the input-output relationship
- SISO case: non-minimal realization leads to zero/pole cancelation in $G(s) = C(sI - A)^{-1}B = \frac{N(s)}{D(s)}$
- MIMO case: may look more complicated....

*May look different from the basic control course....*

# Stabilizability and detectability

- A system is said stabilizable if there exists a matrix $L$ such that $A - BL$ is asymptotically stable.

- A system is said detectable if there exists a matrix $K$ such that $A - KC$ is asymptotically stable.

In other words: a system is

- stabilizable if the unstable eigenvalues of $A$ can be modified using $L$ (i.e., non-controllable eigenvalues are stable)

- detectable if the unstable eigenvalues of $A$ can be modified using $K$ (i.e., non-observable eigenvalues are stable)

# Checking stabilizability and detectability: PBH test

- $(A, B)$ controllable $\iff$ $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full rank for all $\lambda$
  (enough to check for $\lambda =$ eigenvalue)

- $(A, C)$ observable $\iff$ $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full rank for all $\lambda$
  (enough to check for $\lambda =$ eigenvalue)

- $(A, B)$ stabilizable $\iff$ $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full rank for all eigenvalues $\lambda$ s.t. $\mathrm{Re}[\lambda] > 0$

- $(A, C)$ detectable $\iff$ $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full rank for all eigenvalues $\lambda$ s.t. $\mathrm{Re}[\lambda] > 0$

# Change of variables

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is transformed by the change of variable $z = Tx$ to

$$\dot{z} = TAT^{-1}z + TBu$$
$$y = CT^{-1}z + Du$$

*Completely analogous to the basic control course*

Structures of special interest for the transformed matrices

- Diagonal form (or Jordan form)
- Controllability canonical form
- Observability canonical form

# Diagonal form

- If $A$ has all different eigenvalues
- $\implies T$ can be chosen so that $TAT^{-1}$ becomes diagonal.
- $\implies$ Diagonal form:

$$\frac{dx}{dt} = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_n \end{pmatrix} x + \begin{pmatrix} \beta_{11} & \dots & \beta_{1m} \\ \beta_{21} & \dots & \beta_{2m} \\ \vdots & & \vdots \\ \beta_{n1} & \dots & \beta_{nm} \end{pmatrix} u$$

$$y = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{p1} & \dots & \gamma_{pn} \end{pmatrix} x$$

*Completely analogous to the basic control course*

# Controllability canonical form, SISO system

- scalar $u$, scalar $y$

$$G(s) = \frac{c_1 s^{n-1} + \cdots + c_{n-1}s + c_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$\frac{dx}{dt} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} c_1 & c_2 \cdots & c_n \end{pmatrix} x$$

Simple!

# Controllability canonical form, MIMO system

- scalar $u$, vector $y$: simple

$$G(s) = \begin{bmatrix} \frac{c_{11}s^{n-1}+\cdots+c_{1n}}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n} \\ \vdots \\ \frac{c_{p1}s^{n-1}+\cdots+c_{pn}}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n} \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} c_{11} & c_{12} \dots & c_{1n} \\ \vdots & & \vdots \\ c_{p1} & c_{p2} \dots & c_{pn} \end{pmatrix} x$$

- vector $u$, vector $y$: difficult

# Observability canonical form, SISO system

- scalar $u$, scalar $y$: simple

$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$\frac{dx}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} x$$

# Observability canonical form, MIMO

- vector $u$, scalar $y$: simple

$$G(s) = \begin{bmatrix} \frac{b_{11}s^{n-1}+\cdots+b_{n1}}{s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n} & \cdots & \frac{b_{1m}s^{n-1}+\cdots+b_{nm}}{s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n} \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{pmatrix} x + \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ & & \\ b_{n1} & \dots & b_{nm} \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} x$$

- vector $u$, vector $y$: difficult

# What is new for multivariale systems?

A MIMO system differs significanty from a SISO system in the following points:

- Properties of a minimal realization (maybe...)
- When passing from $G(s)$ to a state space form (controllability and observability canonical forms are tricky if you have *simultaneously* multiple inputs and multiple outputs)
- Computing poles in a transfer function matrix
- Computing zeros in a transfer function matrix

# Poles of $G(s)$

Poles:

1. If $(A, B, C, D)$ minimal realization (i.e., controllable and observable) is given:

   - Poles $=$ eigenvalues of $A$

2. If G(s) is given instead?

   - The **pole polynomial** is the least common multiple of all denominators of all minors (i.e., determinant of square submatrices) of $G(s)$.
   - Poles are the roots of the pole polynomial.

Note that in this way we find out the order of a minimal realization!

# Computing poles: examples

**Example 1** : heat exchanger

- Minimal realization:

$$\dot{x} = \begin{bmatrix} -0.21 & 0.2 \\ 0.2 & -0.21 \end{bmatrix} x + \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} u$$
$$y = x$$

- Transfer function

$$G(s) = C(sI - A)^{-1}B = \frac{1}{\det(sI - A)} C \,\mathrm{adj}(sI - A)B$$
$$= \frac{0.01}{(s + 0.01)(s + 0.41)} \begin{bmatrix} s + 0.21 & 0.2 \\ 0.2 & s + 0.21 \end{bmatrix}$$

- Poles: $s = \{-0.01, \, -0.41\}$ = eigenvalues of $A$

# Computing poles: examples

**Example 2**:

- Non-minimal realization:

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- Transfer function (has a cancellation)

$$G(s) = \frac{1}{\det(sI - A)} C \,\mathrm{adj}(sI - A) B$$
$$= \frac{1}{(s+1)\cancel{(s+2)}} \begin{bmatrix} \cancel{s+2} & 2\cancel{(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \end{bmatrix}$$

- Poles: $s = \{-1\} \neq$ eigenvalues of $A = \{-1, -2\}$

# Zeros of $G(s)$

Zeros:

1. If $G$ square:
   - The **zero polynomial** is the numerator of $\det(G)$ (normalized to have the pole polynomial as denominator)
   - Zeros are the roots of the zero polynomial
   - Also: zeros = poles of $G^{-1}(s)$

2. General case:
   - The **zero polynomial** of $G(s)$ is the greatest common divisor for the numerators of the maximal minors of $G(s)$ (normalized to have the pole polynomial as denominator)
   - Zeros are the roots of the zero polynomial

# Computing zeros: examples

**Example 1** : heat exchanger

- Transfer function

$$G(s) = \frac{0.01}{(s+0.01)(s+0.41)} \begin{bmatrix} s+0.21 & 0.2 \\ 0.2 & s+0.21 \end{bmatrix}$$

- $G(s)$ square $\implies \quad \det(G(s)) = \dfrac{0.01}{(s+0.01)(s+0.41)}$
- Zeros: no zeros

**Example 2**:

- Transfer function

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \end{bmatrix}$$

- Greatest common divisor of the numerators of the maximal minors is a constant
- Zeros: no zeros

# Zeros (of a state space model)

State space criterion for zero:

Zeros = values of $s$ for which the matrix

$$M(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} \text{ looses rank}$$

- Criterion gives the same result as the definition based on transfer function (provided one considers a minimal realization)

# Zeros: physical interpretation

Meaning: $z$ is zero if $u = e^{zt}u_o$ leads to $y \equiv 0$ (for appropriate initial value $x(0) = x_o$ and $u_o$) $\implies$ transmission zero

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} u$$

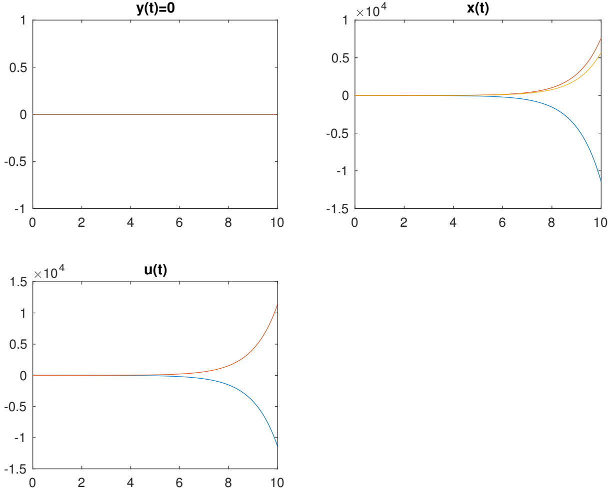
$$y = \begin{bmatrix} 1 & 1.5 & 0 \\ 0.5 & 0 & 1 \end{bmatrix} x$$

has a zero in $z = 1$.

Computing $u_o$ and the initial value $x_o$ which gives $y = 0$, one gets $u = e^{zt}u_o$.

# Zeros: physical interpretation (cont'd)

Let $u(t) = e^{1t}\begin{bmatrix}0.52 & -0.52\end{bmatrix}^T$ and $x_o = \begin{bmatrix}0.52 & -0.35 & -0.26\end{bmatrix}^T$



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