TSRT09 – Control Theory

Lecture 12: Limitations in control design
Course summary

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Summary of Lecture 11

- Control synthesis for nonlinear systems
  - Feedback design by Jacobian linearization
    + nonlinear verification
  - Nonlinear IMC
  - Model predictive control
  - Optimal control
  - Exact linearization

Summary of Lecture 11 (cont'd)

Exact linearization

- Differentiate y until u appears explicitly ($r =$ relative degree)

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x) u$$

- Choose u so that the nonlinearities are canceled in $y^{(r)}$

$$u = \frac{-L_f^r h(x) + v}{L_g L_f^{r-1} h(x)} \quad \Longrightarrow \quad y^{(r)} = v$$

- If relative degree $r = n \implies$ full state linearization
- If relative degree $r < n \implies$ input/output linearization + zero dynamics (which must be stable)

Back to PART II: LINEAR CONTROL THEORY

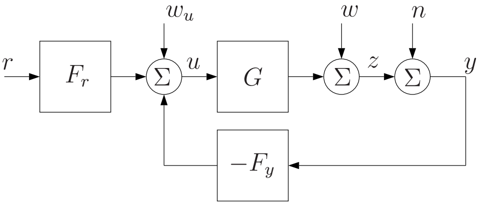
- Lecture 4: The closed-loop system
- Lecture 5: Controller structure
- Lecture 6: LQ-regulation
- Lecture 7: Loopshaping
- Lecture 12: Basic limitations in control design

In the book: Ch. 7

Basic limitations in control design

- Compromise between S and T, Bode relationship
- How small can S become? Bode integral
- Poles in Right Half Plane (RHP)
- Zeros in RHP, delay
- Both poles and zeros in RHP
- Limited input

The closed-loop system: important transfer functions



Loop gain:

$$GF_y$$

Closed-loop transfer function

$$G_c = (I + GF_y)^{-1} GF_r$$

Sensitivity

$$S = (I + GF_y)^{-1}$$

Complementary sensitivity

$$T = (I + GF_y)^{-1} GF_y$$

Compromise between S and T

Ideally, S and T should both be small for all ω, but there is a conflict

$$S + T = I$$

Compromise

- S is small for small frequencies (damped process disturbances; insensitivity to model errors)
- T is small for high frequencies (reduce the effect of measurement noises; stability)

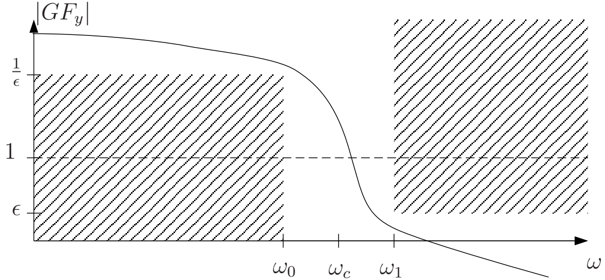
Compromise between S and T

- both S and T can be determined by the loop gain GF_y
- For a small ε it holds approximatively

$$|S| < \varepsilon \iff |GF_y| > \frac{1}{\varepsilon} \qquad\qquad |T| < \varepsilon \iff |GF_y| < \varepsilon$$

(another way of seeing that $|S|$ and $|T|$ cannot be made small at the same frequencies)

How fast can $|GF_y|$ change?



- How fast can you pass from "small S" to "small T"?
- How small can $\omega_1 - \omega_0$ become?
- There is a relationship between amplitude and phase in a transfer function (e.g. GF_y) that prevents us from making an arbitrary fast transition...

Bode relationship: coupling of amplitude and phase

For loop gain $G(i\omega)F_y(i\omega)$: (approximate) Bode relationship

$$\underbrace{\arg(G(i\omega)F_y(i\omega))}_{\text{phase of } GF_y} \lessapprox \frac{\pi}{2} \underbrace{\frac{d}{d\log\omega} \cdot \log|G(i\omega)F_y(i\omega)|}_{\text{Slope of } GF_y \text{ in the Bode diagram}}$$

Interpretation:

- Bode's relationship provides an upper bound on the phase, which depends on the derivative of the amplitude

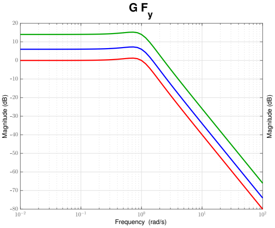
Consequences:

- $|G(i\omega)F_y(i\omega)|$ rapidly decaying means problems with the phase
- Phase around ω_c must be higher than $-180°$ (Nyquist criterion), i.e., the amplitude cannot decrease faster than "slope -2" around ω_c for the system to be stable

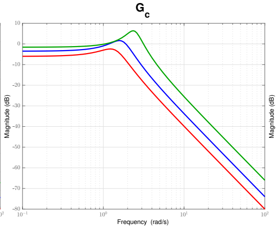
$\Longrightarrow$ around ω_c neiher $|S|$ "small" nor $|T|$ "small" can be satisfied

Limitation on S: example

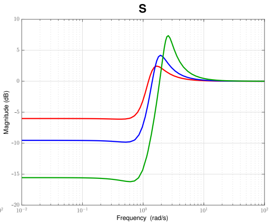
$$G(s) = \frac{1}{s^2 + s + 1} \qquad F_y = K, \quad K = {\color{red}1}, {\color{blue}2}, {\color{green}5}$$



loop gain



closed loop



sensitivity

Even if we neglect $S + T = 1$ we cannot make S arbitrarily small!

Limitations on S: Bode integral

Assume the (SISO) loop gain GF_y is such that $|GF_y|$ decays at least as $|s|^{-2}$ when $|s| \to \infty$

- If GF_y has no poles in RHP

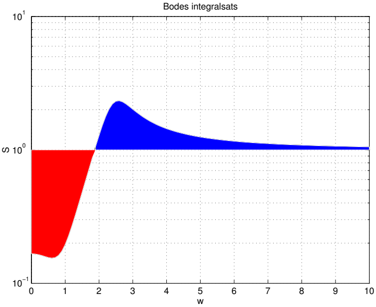
$$\int_0^\infty \log |S(i\omega)| d\omega = 0$$

- If GF_y has M poles in RHP: p_i, $i = 1, \ldots, M$

$$\int_0^\infty \log |S(i\omega)| d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}(p_i)$$

Bode integral: invariance property of S

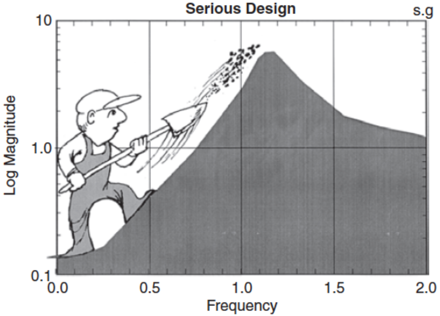
$\int_0^\infty \log |S(i\omega)| d\omega = 0$ is a conserved quantity



Sensitivity $|S(i\omega)| < 1$ at certain frequencies (red) has to be compensated by $|S(i\omega)| > 1$ at other frequencies (blue).

Bode integral: invariance property of S

Loopshaping = "displacing the dirt"



G. Stein, Respect the unstable, IEEE Control Systems Magazine 23(4):12-25, 2003

Bode integral: considerations

- $\int_0^\infty \log|S(i\omega)|d\omega$ is a conserved quantity, no matter what the controller is
- The assumption of decay rate in GF_y ($|s|^{-2}$ for large s) is fulfilled if both $G(s)$ and $F_y(s)$ are strictly proper (physically reasonable)
- Stable systems: the sensitivity cannot be < 1 at all frequencies
- Unstable systems (i.e., GF_y with poles in RHP): the situation gets worse. The regions where $|S(i\omega)| > 1$ dominate. The faster the unstable poles, the worse the situation becomes
- GF_y becomes unstable if F_y unstable $\implies$ avoid unstable controllers!

Systems with poles in RHP

- Requires very reliable controllers. If something goes wrong...
- Bounded signal can stabilize the system only in a part of the state space
- An unstable pole p_1 sets a lower bound on the bandwidth
- Intuition:
  1. open loop response has a (exponentially growing) mode $e^{p_1 t}$
  2. its time constant $\approx \frac{1}{p_1}$
  3. controller must react in a time-scale faster than $\frac{1}{p_1}$
  4. gain crossover $\approx$ bandwidth

$$\omega_c > \approx p_1$$

- More technical argument: constraint on T

$$\omega_c > 2p_1$$

Example: inverted pendulum

Example: inverted pendulum

$$m\ell\ddot{\theta} + \ell f\dot{\theta} + mg\sin\theta = u$$

- state space

$$\begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = u - \underbrace{\tfrac{f}{m}}_{=1} x_2 - \tfrac{g}{\ell}\sin x_1 \end{cases}$$

- equilibrium

$$x_o = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

- saddle point

unstable pole p_1

$$\lambda_{12} = \frac{1}{2}\left(-1 \pm \sqrt{1 + \frac{4g}{\ell}}\right) \quad \Longrightarrow \quad p_1 \simeq \sqrt{\frac{g}{\ell}}$$

Example: epidemic model

Example: epidemic model

$$\frac{dS}{dt} = -\alpha SI$$
$$\frac{dI}{dt} = \alpha SI - \beta I$$
$$\frac{dR}{dt} = \beta I$$

Variables:

- $S =$ susceptible
- $I =$ infected
- $R =$ removed

Parameters:

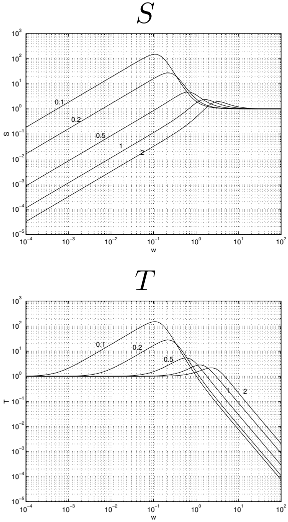
- $\alpha =$ infectivity rate ($1 < \alpha \leq 10$)
- $\beta =$ removal rate ($\beta = 1$)

Assume $S = 1$, neglect R, add a control input u, and measure I

$$\frac{dI}{dt} = \alpha I - \beta I + u$$
$$y = I$$

Example: pole in RHP

$$G(s) = \frac{s+1}{s(s-1)}$$



- Regulator F_y s.t. the closed loop system has (double) poles in $a(-1 \pm i)$, $a = 0.1,\ 0.2,\ 0.5,\ 1,\ 2$
- Bandwidth in T never drops below $\omega = p_1 = 1$ even when the closed loop system is very slow
- Price to pay: both S and T have large peaks when closed loop poles become slow

System with delay

- Loop gain of a system with time delay T_d:

$$G(s)F_y(s) = H(s)e^{-sT_d}$$

- Consequences:
  - controller cannot react in a time scale faster than T_d
  - $\implies$ bandwidth cannot be bigger than $\approx \frac{1}{T_d}$

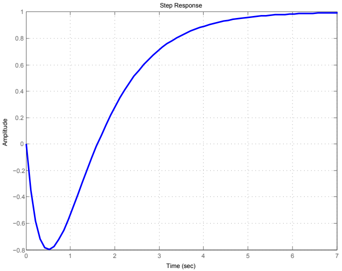
$$\omega_c < \approx \frac{1}{T_d}$$

$\implies$ upper bound on the bandwidth

Systems with zeros in RHP: example

Example: Step response for

$$\frac{-4s+2}{(s+1)(s+2)}$$



"The step response initially starts in the wrong direction"

$\Longrightarrow$ a regulator that reacts too fast amplifies the wrong direction

$\Longrightarrow$ upper bound on the bandwidth

- Intuitive bound:

$$\omega_c \leq z_1$$

- more technical argument: constraint on S

$$\omega_c \leq z_1/2$$

Non-minimum phase systems: example

Step responses for

- zero in RHP

$$\frac{-4s+2}{(s+1)(s+2)}$$

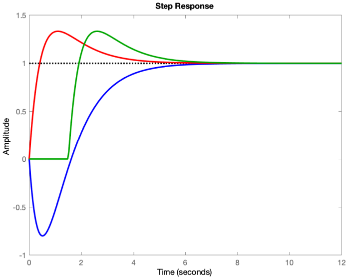
- "equivalent" minimum phase

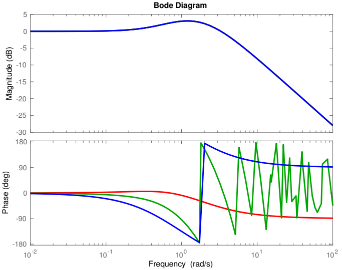
$$\frac{4s+2}{(s+1)(s+2)}$$

- time-delayed

$$\frac{(4s+2)e^{-sT}}{(s+1)(s+2)}$$

Both are examples of non-minimum phase systems





Non-minimum phase systems: considerations

Non minimum-phase systems:

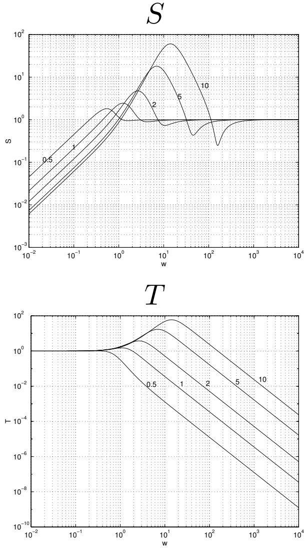
1. Zero in RHP: the system gain for fast changes has the opposite direction w.r.t. that of slow changes
2. Time delay: also in this case the reaction of the regulator cannot be arbitrarily fast

- Non-minimum phase systems have a lower phase margin than their minimum phase counterpart $\implies$ more "stability critical"
- Avoid fast regulator by imposing an upper limit on the bandwidth ω_c

Example: zero in RHP

$$G(s) = \frac{-s+1}{s(s+1)}$$

- Regulator F_y s.t. the closed loop system has (double) poles in $a(-1 \pm i)$, $a = 0.5,\ 1,\ 2,\ 5,\ 10$
- Crossing $|S| = 1$ cannot be pushed higher than $\omega = z_1 = 1$
- Price to pay: high peak in $|S|$ and $|T|$ when system becomes fast
- Bandwidth in T grows as expected



Zeros and feedback

Consider

$$G_c = \frac{GF_r}{1 + GF_y}, \qquad T = \frac{GF_y}{1 + GF_y}$$

If $G(z_1) = 0$, then it is also $G_c(z_1) = 0$ and $T(z_1) = 0$

- Zeros of G are also zeros of G_c and T, i.e., feedback cannot move the zeros
- Zeros (in LHP) can sometimes be canceled (price to pay: loss of controllability and/or observability)
- Zeros in the RHP cannot be canceled, because that requires unstable pole in the regulator F_y and/or F_r
- $\Longrightarrow$ it is impossible to go around the limitations imposed by RHP zeros

Both poles and zeros in RHP

- Rules of thumb:

$$\omega_c > 2p_1 \qquad\qquad \omega_c < \frac{z_1}{2}$$

- If unstable pole lies to the left of the non-minimum phase zero $\Longrightarrow$ can be controlled (rules above can be compatible)

- If the unstable pole lies to the right on the non-minimum phase zero in the RHP
  $\Longrightarrow$ very difficult to control! (conflicting rules of thumb)

Example: both zero and pole in RHP

$$G(s) = \frac{10(-s+1)}{s(s-10)}$$

- Regulator F_y s.t. the closed loop system has (double) poles in $a(-1 \pm i)$, $a = 1,\ 3,\ 10$
- Pole in RHP: $p_1 = 10$
- Zero in RHP: $z_1 = 1$

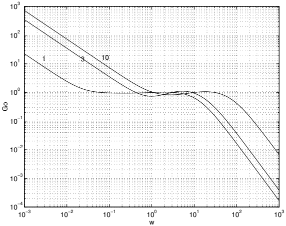
$\Longrightarrow$ worst case scenario: incompatible rules of thumb

$$\begin{aligned} \text{pole in RHP} \quad &\Longrightarrow \quad \omega_c > 20 \text{ rad/s} \\ \text{zero in RHP} \quad &\Longrightarrow \quad \omega_c < 0.5 \text{ rad/s} \end{aligned}$$

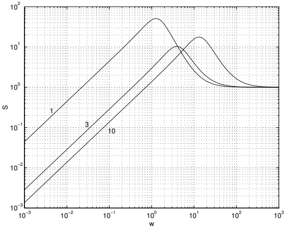
Example: both zero and pole in RHP (cont'd)

- Solution: shape the loop gain to be close to 1 in the interval $[0.5,\ 20]$ rad/s
- Both S and T are bad in the interval $[0.5,\ 20]$ rad/s

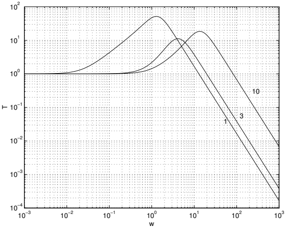
GF_y



S



T



Example: epidemic model

- Add a delay (incubation time)

$$\begin{aligned} \frac{dI}{dt} &= \gamma I + u \\ y &= I(t - T_d) \end{aligned} \qquad \Longrightarrow \qquad G(s)e^{-T_d s} = \frac{e^{-sT_d}}{s-\gamma}$$

- Use a Padé approximation: $e^{-sT_d} \simeq \frac{-\frac{T_d}{2}s+1}{\frac{T_d}{2}s+1}$

$$G(s)e^{-T_d s} \simeq \frac{-\frac{T_d}{2}s+1}{(\frac{T_d}{2}s+1)(s-\gamma)}$$

- $\Longrightarrow$ System with zero and pole in RHP

$$z = \frac{2}{T_d} \qquad p = \gamma$$

Example: epidemic model

$$G(s)e^{-T_d s} \simeq \frac{-\frac{T_d}{2}s + 1}{(\frac{T_d}{2}s + 1)(s - \gamma)}$$

- zero and pole in RHP

$$z = \frac{2}{T_d} \sim 4 \qquad \Longrightarrow \qquad \text{easy}$$
$$p = \gamma \sim 1 \qquad\qquad\qquad p < z$$

- Can it be stabilized?

Example: epidemic model

$$G(s)e^{-T_d s} \simeq \frac{-\frac{T_d}{2}s + 1}{(\frac{T_d}{2}s + 1)(s - \gamma)}$$

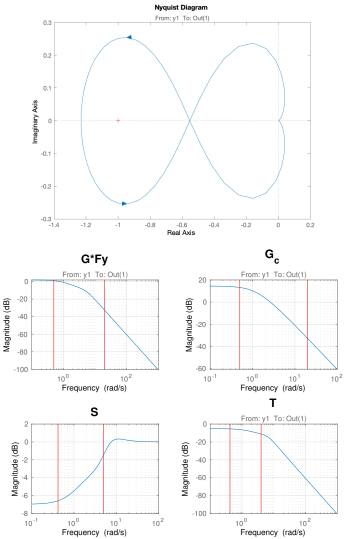
- zero and pole in RHP

$$z = \frac{2}{T_d} \sim 4 \qquad \Longrightarrow \qquad \text{easy}$$

$$p = \gamma \sim 1 \qquad\qquad\qquad p < z$$

- Can it be stabilized?
  yes (with stable controller)

$$F_y = \frac{-9.2031(s + 4.004)}{(s^2 + 7.981s + 29.92)}$$



Example: epidemic model

$$G(s)e^{-T_d s} \simeq \frac{-\frac{T_d}{2}s + 1}{(\frac{T_d}{2}s + 1)(s - \gamma)}$$

- zero and pole in RHP

$$z = \frac{2}{T_d} \sim 0.2 \qquad \Longrightarrow \qquad \text{difficult}$$

$$p = \gamma \sim 4 \qquad\qquad z < p$$

- Can it be stabilized nevertheless?

Example: epidemic model

$$G(s)e^{-T_d s} \simeq \frac{-\frac{T_d}{2}s + 1}{(\frac{T_d}{2}s + 1)(s - \gamma)}$$

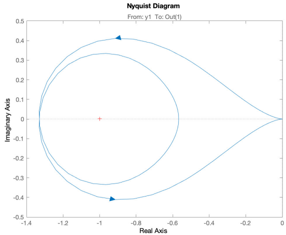
- zero and pole in RHP

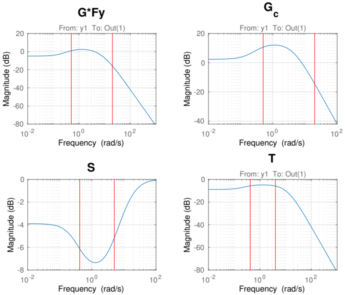
$$z = \frac{2}{T_d} \sim 0.2 \qquad \Longrightarrow \qquad \text{difficult}$$

$$p = \gamma \sim 4 \qquad\qquad z < p$$

- Can it be stabilized nevertheless? yes! (with unstable controller)

$$F_y = \frac{79(s + 0.1877)}{(s + 13.26)(s - 0.4925)}$$





Poles/zeros in the RHP, summary

- Pole in the RHP
  - Lower bound for the bandwidth
  - Harder when the pole is moved to the right
  - High operational reliability requirements
- Zero in the RHP
  - Upper bound on the bandwidth
  - Harder when the zero moves to the left
- Both pole and zero in the RHP
  - Pole to the left of the zero: can be OK
  - Pole to the right of the zero: very difficult to control

Bounds on the control signal: example

On 24th November 2004 the passenger ferry Casino Express was grounded while entering the port of Umeå due to high winds

Crash investigation:

- The wind power on the upper parts was at least 600 kN (20 m/s wind speed)
- No combination of control signals (propellers, rudders) could have compensated that
- Not even tug assistance (max 260 kN) was enough

When can a control signal compensate for disturbances?

Control signal u, disturbance d:

$$z = G(s)u + G_d(s)d$$

for some G, G_d (scalar, for simplicity)

- Assume $|u(t)| \leq u_0$ and $|d(t)| \leq d_0$
- Then it must hold

$$u_0 \geq \frac{|G_d(i\omega)|}{|G(i\omega)|} d_0, \quad \forall\ \omega$$

  if d is to be perfectly eliminated
- If this is not fulfilled, then no controller (linear or nonlinear) can perfectly compensate the disturbance

The end!

This is the end!

… but not before a few final observations

Summary of the course

1. Part I: Linear systems
   - Multivariable systems
   - Disturbance description
   - Kalman filter

2. Part II: Linear control theory
   - Closed-loop system description; controller structure
   - LQG-design
   - Loopshaping: $\mathcal{H}_2$, $\mathcal{H}_\infty$ methods
   - Specifications and limitations in control design

3. Part III: Nonlinear control theory
   - Nonlinearities and nonlinear stability
   - Self-sustained oscillations
   - Exact linearization

Summary of the course: ''Take home messages''

- What you should do as a control engineer:

  1. Start always testing a simple regulator first
  2. If unsatisfied (and if you are confident in your model) use the advanced methods you learned ... keeping in mind that

     1. there are some fundamental limitations to what one can achieve
     2. there is no "automatic control design" in automatic control! (designing a controller is never just pushing a button on a toolbox)

- Some of the most important successes of control: apply feedback where nobody had though about using it before!

Summary of the course: ”Take home messages”

- Many of the methods your learned in this course are applicable also outside control theory
- Analysis methods
  - model building
  - signal modeling
  - Kalman filter
  - linearization
  - stability
  - Lyapunov function
  - phase plane ....
- System-thinking: systematic, analytical thinking about dynamics

Summary for exam

- Exam: in the ISY computer rooms

- Content:
  - 4-5 exercises, some to be solved with Matlab (submit printout of relevant code / plots)
  - all chapters covered in the lectures and exercises (see course home page) can be exam material

- Help material
  - Tabels, Calculator
  - Matlab
  - Control Theory book, Basic control book (grundkursboken)
  - No lab or exercise material, "Math primer", or old exams

Tack!

Lycka till!

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