TSRT09 – Control Theory

Lecture 11: Nonlinear control design and exact linearization

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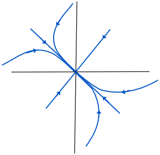
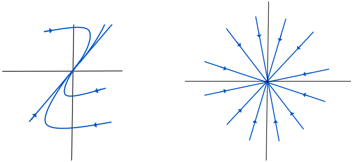
Summary of lecture 10: Phase plane

- Node, distinct eigenvalues



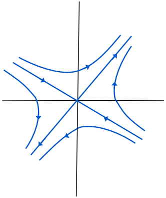
(2 real λ_i, same sign)

- Node, identical eigenvalues



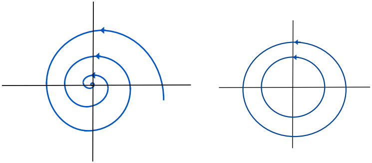
(1 or 2 eigenvectors)

- Saddle point



(2 real λ_i, different sign)

- Focus and center



(complex conj. eigenvalues)

Summary of lecture 10: nonlinear vs linearization

Linearized system

$$\dot{x} = Ax$$

has equilibrium point

1. node, focus or saddle point

2. center

3. continuum of equilibria $\lambda_1 = 0,\ \lambda_2 \neq 0$

Nonlinear system

$$\dot{x} = f(x)$$

has equilibrium point

1. decidable: same type

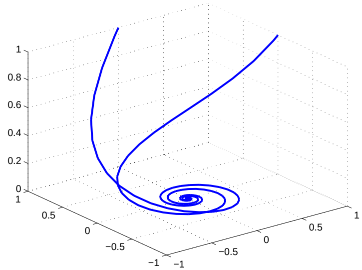
2. undecidable: can be either a (stable/unstable) focus or a center

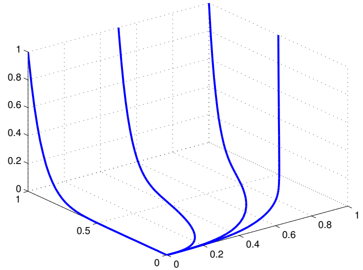
3. undecidable: can be a (stable/unstable) node, or saddle point or a continuum of equilibria

Two examples with three state variables

Example of generalization to higher dimension:

- Stable "focus node" (left)
  - Focus + one real eigenvalue
- Stable equilibrium with 3 distinct real eigenvalues (right)
  - Generalization of 2 distinct real eigenvalues case
- Plus all combinations of these....





Lecture 11

- Control synthesis for nonlinear systems
  - Feedback design by Jacobian linearization
    + nonlinear verification
  - Nonlinear IMC
  - Model predictive control
  - Optimal control
  - Exact linearization

In the book: Ch. 15, 17

Feedback design by Jacobian linearization: state feedback

- Nonlinear system

$$\dot{x} = f(x, u)$$

- Equilibrium point (x_0, u_0):

$$0 = f(x_0, u_0)$$

- Jacobian linearization: $z = x - x_0, \quad v = u - u_0$

$$\dot{z} = Az + Bv$$

where Jacobian

$$A = \frac{\partial f(x_0, u_0)}{\partial x}, \quad B = \frac{\partial f(x_0, u_0)}{\partial u}$$

Feedback design by Jacobian linearization: state feedback

State feedback

- If $(A,\ B)$ controllable (or stabilizable) $\implies$ can choose L such that the feedback $v = -Lz$ renders the closed loop linearization

$$\dot{z} = (A - BL)z$$

asymptotically stable

- $\implies$ equilibrium (x_0, u_0) is locally asymptotically stable for the original nonlinear system with feedback

$$u = u_0 - L(x - x_0)$$

Example: phase plane for generator/pendulum

System:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_2 - b\sin x_1 + u$$

- upward equilibrium: $x_0 = \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \quad u_0 = 0$
- linearization

$$A = \begin{bmatrix} 0 & 1 \\ b & -a \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- eigenvalues:

$$\lambda_{1,2} = (-a \pm \sqrt{a^2 + 4b})/2, \qquad \lambda_1 < 0, \quad \lambda_2 > 0$$

- $\implies$ saddle point

Feedback design by Jacobian linearization: output feedback

- Nonlinear system with output

$$\dot{x} = f(x, u)$$
$$y = h(x)$$

- Equilibrium point (x_0, u_0, y_0):

$$0 = f(x_0, u_0)$$
$$y_0 = h(x_0)$$

- Linearization: $z = x - x_0$, $v = u - u_0$, $\xi = y - y_0$

$$\dot{z} = Az + Bv$$
$$\xi = Cz$$

where Jacobians

$$A = \frac{\partial f(x_0, u_0)}{\partial x}, \quad B = \frac{\partial f(x_0, u_0)}{\partial u}, \quad C = \frac{\partial h(x_0)}{\partial x}$$

Feedback design by Jacobian linearization: output feedback

Output feedback

- If (A, B) controllable (or stabilizable)
  (A, C) observable (or detectable)
- $\Longrightarrow$ can use observer-based feedback to locally asymptotically stabilize the original nonlinear feedback

$$\dot{\hat{z}} = A\hat{z} + Bv + K(\xi - C\hat{z})$$
$$v = -L\hat{z}$$

  where
  - K s.t. $(A - KC)$ is asymptotically stable
  - L s.t. $(A - BL)$ is asymptotically stable
- Note: not a Kalman filter!
- Separation principle does not hold (globally)

Example: phase plane for generator/pendulum

System with output:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - b\sin x_1 + u \\ y &= x_1\end{aligned}$$

- upward equilibrium: $x_0 = \begin{bmatrix}\pi \\ 0\end{bmatrix}, \quad u_0 = 0$
- linearization

$$A = \begin{bmatrix}0 & 1 \\ b & -a\end{bmatrix}, \quad B = \begin{bmatrix}0 \\ 1\end{bmatrix}, \quad C = \begin{bmatrix}1 & 0\end{bmatrix}$$

Feedback design by Jacobian linearization: output feedback

Alternative: nonlinear observer

- observer = replica of the system
- if the system is known, why not use it as an observer?

$$\dot{\hat{x}} = f(\hat{x}, u) + K(y - h(\hat{x}))$$
$$u = -L\hat{x}$$

- observer gain K
  - computed based on linearization at x_0
  - computed the linearization at each $\hat{x}$ (“extended Kalman filter”)

Linear design, nonlinear verification

- Choose an equilibrium point
- Linearize the dynamics at that equilibrium
- Use linear methods (e.g. LQR/LQG) to design a linear regulator for the linearized system
- Simulate the nonlinear system with the linear regulator. Verify that it works properly
- In case, use nonlinear analysis (e.g. describing function) to check that the nonlinearity does not give problems

Optimal control

Minimize

$$\int_0^\infty L(x,u)dt$$

for the system

$$\dot{x} = f(x,u)$$

- Powerful method for computing a controller
- In some special cases: analytical solution; in all others: numerical
- Often computationally intensive

To know more: TSRT08 Optimal Control

Goddards rocket problem

Model predictive control (MPC)

For many applications: modified quadratic problem

$$\min \sum_{0}^{T} (x^T Q_1 x + u^T Q_2 u)\, dt$$
$$x(t+1) = Ax(t) + Bu(t)$$
$$|u_i(t)| \le a_i, \quad |x_j(t)| \le b_j$$

MPC: two simplifications in the minimization:

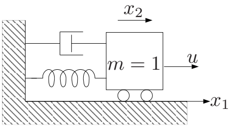
- Optimization is over a finite (“moving”) horizon
- Dynamics is discretized $\Longrightarrow$ constant input on each time interval
- Can work (in principle) also for nonlinear system

To know more:
TSRT07 Industrial Control, TSRT08 Optimal Control

Exact linearization: mechanical system examples

x_1 = position, x_2 = velocity:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -k(x_1) - \ell(x_2) + u$$



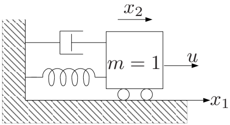
where

- $k(x_1)$ is a nonlinear position-dependent force/torque (spring force, gravitation, etc)
- $\ell(x_2)$ is a nonlinear velocity-dependent force/torque (damping, friction)
- u input

Exact linearization: mechanical systems

x_1 = position, x_2 = velocity:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -k(x_1) - \ell(x_2) + u$$



Change of input that directly compensate for the nonlinearities (computer-controlled torque):

$$u = k(x_1) + \ell(x_2) + v$$

$\Longrightarrow$ linear system (with new "virtual" input v)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = v$$

$\Longrightarrow$ System is exactly linearized! $\rightarrow$ feedback linearization

Exact linearization: cruise control in a airplane

$x_1 =$ velocity, $x_2 =$ engine thrust:

$$\dot{x}_1 = -p(x_1) + x_2$$
$$\dot{x}_2 = -q(x_2) + u$$

where

- $p(x_1) =$ air drag
- $q(x_2) =$ damping term
- $u =$ input

No longer possible to use exact linearization method in a straightforward way...

Exact linearization

Control affine nonlinear SISO system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{aligned}
$$

- Task: obtain explicit expression for the dependence of y from u
- Approach: differentiate the output

$$
\begin{aligned}
\dot{y} &= \frac{\partial h}{\partial x}\big(f(x) + g(x)u\big) \\
&= \underbrace{\sum_{i=1}^{n} f_i(x)\frac{\partial h}{\partial x_i}}_{L_f h(x)} + \underbrace{\sum_{i=1}^{n} g_i(x)\frac{\partial h}{\partial x_i}}_{L_g h(x)} u = \quad L_f h(x) + L_g h(x) u
\end{aligned}
$$

- where $L_f h(x) =$ Lie derivative of $h(x)$ along vector field f

Exact linearization

$$\dot{y} = L_f h(x) + L_g h(x) u$$

- If $L_g h(x) = 0$, differentiate again

$$\begin{aligned}\ddot{y} =& \frac{\partial L_f h(x)}{\partial x}\big(f(x) + g(x)u\big) \\ =& L_f^2 h(x) + L_g L_f h(x) u\end{aligned}$$

- If $L_g L_f h(x) = 0$, keep differentiating
- ....
- Stop differentiating when $L_g L_f^{r-1} h(x) \neq 0$

$$y^{(r)} = L_f^r h(x) + \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0} u$$

- Relative degree $= r$

Exact linearization

$$y^{(r)} = L_f^r h(x) + \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0} u$$

- Change of input

$$u = \frac{-L_f^r h(x) + v}{L_g L_f^{r-1} h(x)}$$

- Input-output linear system

$$y^{(r)} = v$$

Case 1: full state exact linearization

Case 1: $r = n$

- Change of state

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

- Change of input

$$u = \frac{-L_f^n h(x) + v}{L_g L_f^{n-1} h(x)}$$

Case 1: full state exact linearization

- New state space model: linear companion form

$$
\begin{aligned}
\dot{z} &= \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & 1 \\ & & & & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} v \ = \ Az + Bv \\
y &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} z
\end{aligned}
$$

- (A, B) always controllable

$\Longrightarrow$ linear feedback $v = -Lz$ achieves stability

Case 2: partial state exact linearization

Case 2: $r < n$

- Change of basis

$$\begin{bmatrix} z_1 \\ \vdots \\ z_r \\ z_{r+1} \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} h(x) \\ \vdots \\ L_f^{r-1}h(x) \\ \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{bmatrix}$$

- New state space model: linear companion form + zero dynamics

- Change of input

$$u = \frac{-L_f^r h(x) + v}{L_g L_f^{r-1} h(x)}$$

linear
$$\begin{cases} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_r &= v \end{cases}$$

zero dynamics
$$\begin{cases} \dot{z}_{r+1} &= \psi_1(z, v) \\ &\vdots \\ \dot{z}_n &= \psi_{n-r}(z, v) \end{cases}$$

$$y = z_1$$

Case 2: partial state linearization

- Linear feedback

$$v = -L \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$$

  stabilizes "input-output part"

- Zero dynamics:

$$\begin{aligned} \dot{z}_{r+1} &= \psi_1(z, v) \\ &\vdots \\ \dot{z}_n &= \psi_{n-r}(z, v) \end{aligned}$$

- y and zero dynamics $z_{r+1} \dots z_n$ are "decoupled"
  $\Longrightarrow$ zero dynamics is not observable from y

- To have "internal" stability: zero dynamics must be stable!
  (minimum phase zero dynamics)

Exact linearization: cruise control in a airplane

$x_1 =$ velocity, $x_2 =$ engine thrust:

$$\dot{x}_1 = -p(x_1) + x_2$$
$$\dot{x}_2 = -q(x_2) + u$$

where

- $p(x_1) =$ air drag
- $q(x_2) =$ decay term
- $u =$ input

A first choice of output:

$$y = x_2$$

$\Longrightarrow$ partial state linearization

Exact linearization: cruise control in a airplane

$x_1 =$ velocity, $x_2 =$ engine thrust:

$$\dot{x}_1 = -p(x_1) + x_2$$
$$\dot{x}_2 = -q(x_2) + u$$

where

- $p(x_1) =$ air drag
- $q(x_2) =$ decay term
- $u =$ input

A better choice of output:

$$y = x_1$$

$\Longrightarrow$ full state linearization

Exact linearization: summary

- Method uses
  1. (nonlinear) change of state $z = \Phi(x)$
  2. (nonlinear) change of input $u = \frac{-L_f^r h(x)+v}{L_g L_f^{r-1} h(x)}$

  $\Longrightarrow$ more powerful than just changing basis!

- If relative degree $r = n$: easy case!
  $\Longrightarrow$ full state exact linearization
  $\Longrightarrow$ controller design is straightforward!

- If relative degree $r < n$: more difficult case
  $\Longrightarrow$ partial state linearization (input-output exact linearization)
  $\Longrightarrow$ must make sure that the zero dynamics is stable

- When no output is given: making a "smart" choice of output can help a lot!

Exact linearization: final comments

- (+) Mathematically elegant
- (-) Possible only for some classes of systems
- (-) Often calculations are complicated
- (-) Adding constraints on the input is difficult
- (-) Robustness is difficult to analyze
- (+) Numerous successful applications

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