TSRT09 – Control Theory

Lecture 10: Phase plane analysis

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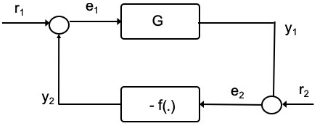
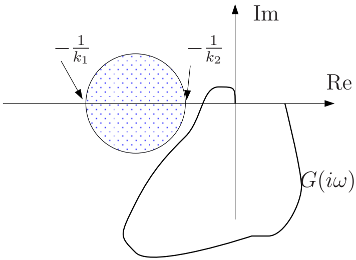
Summary of lecture 9. Circle criterion

Stable linear system $G(s)$ feedback coupled with a static sector nonlinearity $f(x)$

$$f(0) = 0, \qquad k_1 x \leq f(x) \leq k_2 x$$

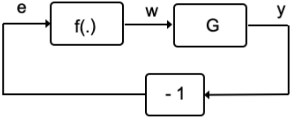
Stability if the Nyquist curve of $G(i\omega)$ does not encircle or enter the circle.





Summary of lecture 9. Describing functions

- Seek self-sustained oscillations in the following system:

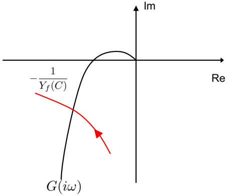


- f represented as an amplitude-dependent gain:
  $\Longrightarrow$ describing function

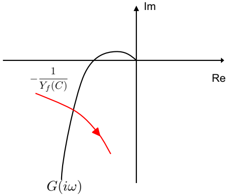
$$Y_f(C) = \frac{A(C)e^{i\phi(C)}}{C} \qquad (C = \text{amplitude})$$

- Condition for oscillations: $G(i\omega) = -\frac{1}{Y_f(C)}$
- Graphical representation of the condition: intersection between Nyquist curve $G(i\omega)$ and $-1/Y_f(C)$
- Method is only approximative

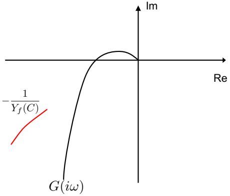
Summary of lecture 9. Amplitude stability of oscillations



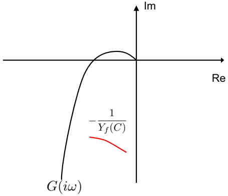
Stable oscillation



Unstable oscillation



Vanishing oscillation



Unbounded growing oscillations

Lecture 10

Phase plane analysis

- Linear phase plane
- Nonlinear phase plane

In the book: Ch. 13

Phase plane

- Phase space: old name for state space
- Phase plane: 2-dimensional state space
- **Phase portrait**: graphical way to describe phase plane
  - Often we simulated $x_1(t)$ and $x_2(t)$ separately as functions of time; here instead they are represented as a single curve
  - Can be sketched using eigenvalues and eigenvectors of matrix A
  - Alternatively, it can be simulated from a number of initial conditions
- Part of the results obtained in dimension 2 can be generalized to higher dimension

Linear system. Eigenvectors

- Consider a system without input:

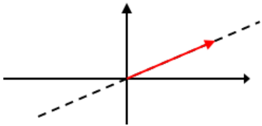
$$\dot{x} = Ax$$

- If λ is an eigenvalue and v an eigenvector of A it holds:

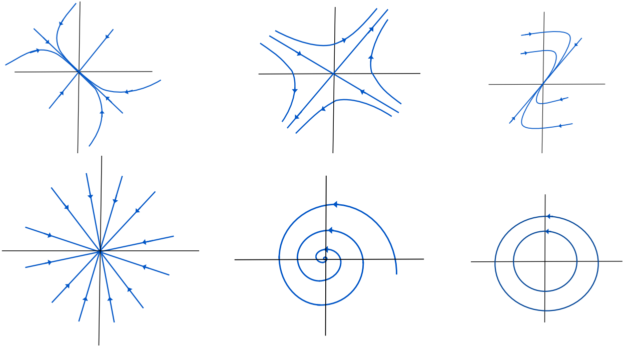
$$Av = \lambda v$$

- If $x(0) = \alpha v \implies$ solution is given by:

$$x(t) = \alpha e^{\lambda t} v$$



Classification of equilibrium points in phase plane

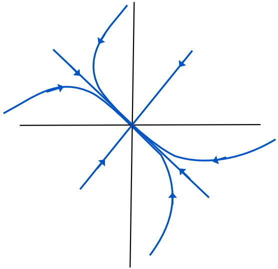


The phase portrait near an equilibrium point depends on whether the eigenvalues are real or complex, and on the sign of the real part

Distinct real eigenvalues: node

- Two real, distinct and negative eigenvalues

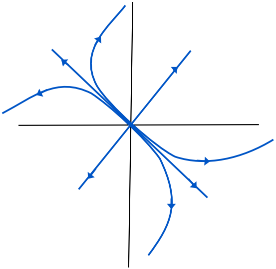
$$\lambda_1 < \lambda_2 < 0$$



Stable

- Two real, distinct and positive eigenvalues

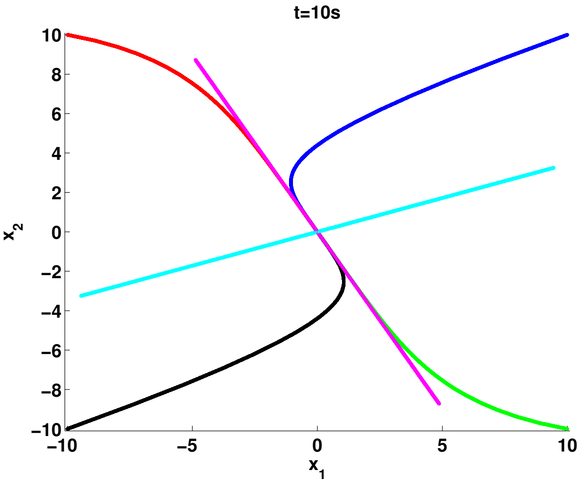
$$0 < \lambda_2 < \lambda_1$$



Unstable

Eigenvector and their ratio determine the phase portrait

Distinct real eigenvalues: node (cont'd)

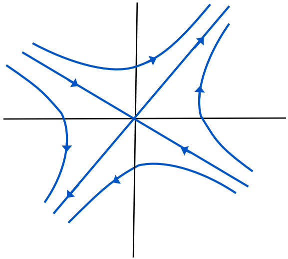


- "fast eigenvector": $\lambda_1 = -2.4$
- "slow eigenvector": $\lambda_2 = -0.52$

Distinct real eigenvalues: : saddle point

- Two real eigenvalues with different sign

$$\lambda_1 < 0 < \lambda_2$$



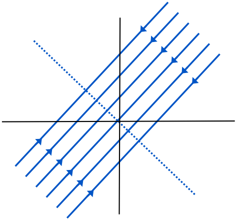
- Stable subspace: if (and only if) initial condition is exactly on the stable eigenspace ($x(0) = \alpha v_1$) then solution is stable

Continuum of equilibria

- One zero, one non-zero eigenvalues:

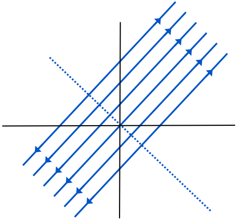
$$\lambda_1 = 0, \qquad \lambda_2 \neq 0 \ \text{ real}$$

$\lambda_2 < 0$



Marginally stable

$\lambda_2 > 0$

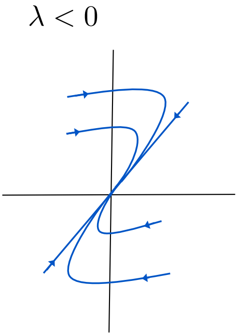


Unstable

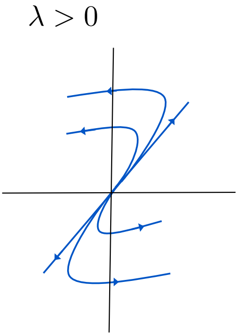
Identical eigenvalues: node

- Identical eigenvalues: both real $\lambda_1 = \lambda_2 = \lambda$

(1) Jordan form $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \implies$ only 1 eigenvector



Stable node

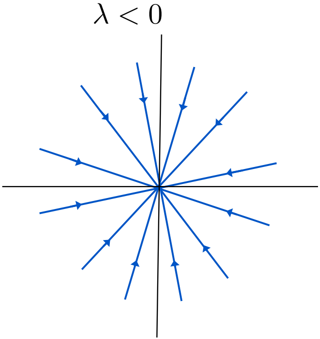


Unstable node

Identical eigenvalues: star node

- Identical eigenvalues: both real $\lambda_1 = \lambda_2 = \lambda$

(2) Jordan form $J = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \Longrightarrow 2$ independent eigenvectors



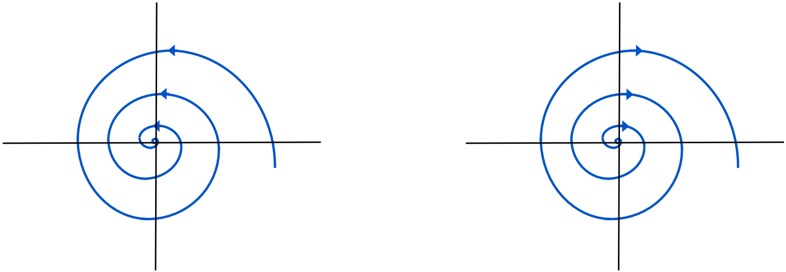
Stable star node

Unstable star node

Complex conjugate eigenvalues: focus

- Complex conjugate eigenvalues $\lambda_{1,2} = \sigma \pm i\omega$

$\sigma < 0$ $\sigma > 0$



Stable focus Unstable focus

Complex conjugate eigenvalues: focus

- Typical form of Jordan block

$$\dot{x} = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} x$$

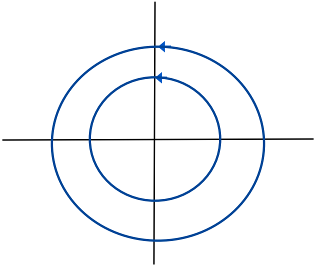
- Alternative form: polar coordinates

$$\begin{cases} x_1 & = r\cos\phi \\ x_2 & = r\sin\phi \end{cases} \quad \Longrightarrow \quad \begin{cases} \dot{r} & = \sigma r \\ \dot{\phi} & = \omega \end{cases}$$

- Solution is a spiral since ϕ grows ($\omega > 0$ imaginary part)
- Distance to the origin
  1. decreases if $\sigma < 0 \implies$ asympt. stability
  2. grows if $\sigma > 0 \implies$ instability

Purely imaginary eigenvalues: center

- Purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega$



Marginally stable

- In polar coordinates

$$\begin{cases} \dot{r} &= \sigma r = 0 \\ \dot{\phi} &= \omega \end{cases} \quad \Longrightarrow \quad r = \text{const}$$

- If $\sigma = 0$ distance to the origin is constant, and the solution is a circle around the origin

Relationship between linear and nonlinear system (near equilibrium point)

Consider nonlinear system

$$\dot{x} = f(x)$$

1. Equilibrium point (here: $x_o = 0$)
2. Linearization near $x_o = 0$

$$\dot{x} = Ax + g(x), \qquad \frac{|g(x)|}{|x|} \xrightarrow{x \to 0} 0$$

3. Check stability of linearization
4. Compute local phase portraint near $x_o = 0$

Relationship between linear and nonlinear system (near equilibrium point)

Linearized system

$$\dot{x} = Ax$$

has equilibrium point

1. node, focus or saddle point
2. center
3. continuum of equlibria $\lambda_1 = 0,\ \lambda_2 \neq 0$

Nonlinear system

$$\dot{x} = f(x)$$

has equilibrium point

1. decidable: same type
2. undecidable: can be either a (stable/unstable) focus or a center
3. undecidable: can be a (stable/unstable) node, or saddle point or a continuum of equilibria

Undecidable case: example

Example

$$\dot{x}_1 = x_2 - x_1^3 \qquad \dot{x}_1 = x_2 + x_1^3$$

$$\dot{x}_2 = -x_1 - x_2^3 \qquad \dot{x}_2 = -x_1 + x_2^3$$

Phase portrait away from equilibrium points

- Nonlinear second order system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

- Construct the ratio of derivates

$$\frac{dx_2}{dx_1} = \frac{dx_2}{dt}\frac{dt}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

- slope for (x_1, x_2) is
  - horizontal when $f_2(x_1, x_2) = 0$
  - vertical when $f_1(x_1, x_2) = 0$
- phase plane behavior from the limit values

$$\lim_{x_1 \to \pm\infty} \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}, \qquad \lim_{x_2 \to \pm\infty} \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

Example: phase plane for generator/pendulum

System:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_2 - b\sin x_1$$

- equilibria: $x_0 = \begin{bmatrix} \pm 2k\pi \\ 0 \end{bmatrix}$
- linearization

$$A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$$

- eigenvalues:

$$\lambda_{1,2} = (-a \pm \sqrt{a^2 - 4b})/2$$
$$= \sigma \pm i\omega$$

- $\implies$ stable focus

- equilibria: $x_0 = \begin{bmatrix} \pi \pm 2k\pi \\ 0 \end{bmatrix}$
- linearization

$$A = \begin{bmatrix} 0 & 1 \\ b & -a \end{bmatrix}$$

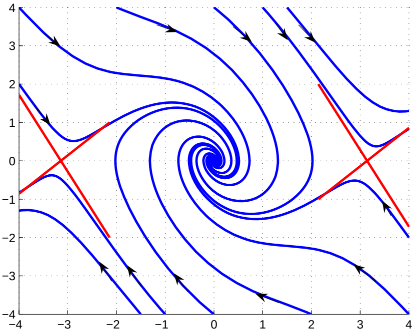
- eigenvalues:

$$\lambda_{1,2} = (-a \pm \sqrt{a^2 + 4b})/2$$

$$\lambda_1 < 0, \quad \lambda_2 > 0$$

- $\implies$ saddle point

Example: phase plane for generator/pendulum



- Equilibrium point in the origin is a **stable focus**
- Equilibrium points in $(\pm\pi, 0)$ are **saddle points** (red lines show eigenvectors)

Example: epidemic model

Example: epidemic model

$$\frac{dS}{dt} = -\alpha SI$$
$$\frac{dI}{dt} = \alpha SI - \beta I$$
$$\frac{dR}{dt} = \beta I$$

Variables:

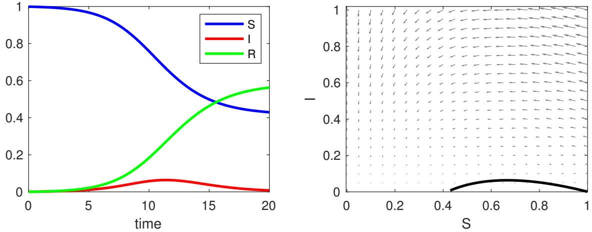
- $S =$ susceptible
- $I =$ infected
- $R =$ removed

Parameters:

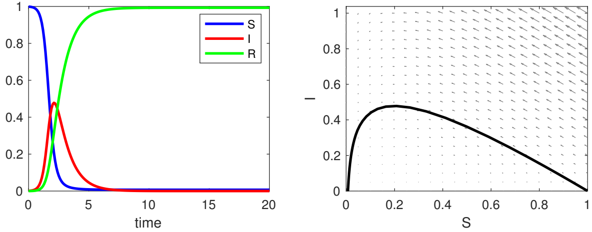
- $\alpha =$ infectivity rate ($1 < \alpha \leq 10$)
- $\beta =$ removal rate ($\beta = 1$)

Example: epidemic model

- low infectivity rate: $\alpha = 1.5$



- high infectivity rate: $\alpha = 5$



Example: system with friction

System

$$m\ddot{z} = u + F$$

- friction

$$F = \begin{cases} -F_1 \text{sign}(\dot{z}) & \text{if } \dot{z} \neq 0 \\ -u & \text{if } \dot{z} = 0 \text{ and } |u| \leq F_o \\ -F_o \text{sign}(u) & \text{if } \dot{z} = 0 \text{ and } |u| > F_o \end{cases}$$

- task: follow constant velocity reference $r(t) = v_o t$
- state variables

$$x_1 = \text{ error } = r - z$$

$$x_2 = \text{ velocity error } = \dot{x}_1 = v_o - \dot{z}$$

Example: system with friction

- feedback (PD controller from error $x_1 = r - z$)

$$u = K_p \underbrace{(r - z)}_{x_1} + K_d \underbrace{(v_o - \dot{z})}_{x_2}$$

- closed loop state space model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}(-u - F) = \frac{1}{m}(-K_p x_1 - K_d x_2 - F) \end{aligned}$$

- parameters: $v_o = 1$, $m = 1$, $F_o = 1.5$, $F_1 = 1$, $K_p = 1$
- different systems in different region of phase plane

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \begin{cases} -x_1 - K_d x_2 + \operatorname{sign}(1 - x_2) & \text{if } x_2 \neq 1 \\ 0 & \text{if } x_2 = 1,\ |x_1 + K_d| \leq 1.5 \\ -x_1 - K_d x_2 + 1.5 \operatorname{sign}(x_1 + K_d) & \text{if } x_2 = 1,\ |x_1 + K_d| > 1.5 \end{cases}$$

Example: system with friction

3 important regions in plane with 3 different linear systems

$\Longrightarrow$ piece-wise linear system: : switching between linear modes

(a) if $x_2 > 1$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - K_d x_2 - 1$$

(b) if $x_2 < 1$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - K_d x_2 + 1$$

(c) if $x_2 = 1$ and $|x_1 + K_d| \leq 1.5$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 0$$

(d) if $x_2 = 1$ and $|x_1 + K_d| > 1.5$
(not important: immediately switches to (a) or (b))

Example: system with friction

- Equilibria analysis
  (a) if $x_2 > 1$
  $$\begin{cases} x_1 & = -1 \\ x_2 & = 0 \end{cases} \implies \text{the equil is outside the region itself}$$
  (b) if $x_2 < 1$
  $$\begin{cases} x_1 & = 1 \\ x_2 & = 0 \end{cases}$$
  (c) if $x_2 = 1$ and $|x_1 + K_d| \leq 1.5 \implies$ no equilibrium

- Linear system (for (a) and (b))

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -K_d \end{bmatrix} \implies \lambda_{1,2} = -\frac{K_d}{2} \pm \frac{1}{2}\sqrt{K_d^2 - 4}$$

Example: system with friction

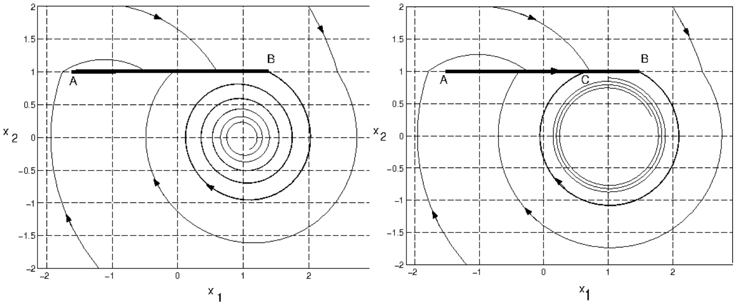
$$\lambda_{1,2} = -\frac{K_d}{2} \pm \frac{1}{2}\sqrt{K_d^2 - 4}$$

Possible cases for the equilibrium in (a) and (b)

1. $K_d = 0$ (i.e. P controller): $\lambda_{1,2} = \pm i \implies$ center
2. $0 < K_d < 2:$ $\lambda_{1,2}$ complex conjugate $\implies$ stable focus
3. $K_d = 2:$ $\lambda_{1,2} = -\frac{K_d}{2} \implies$ real, negative eigenval of multipl. 2
4. $K_d > 2:$ $\lambda_{1,2} \implies$ two distinct, real, negative eigenvalues
   $\implies$ stable node

Example: system with friction

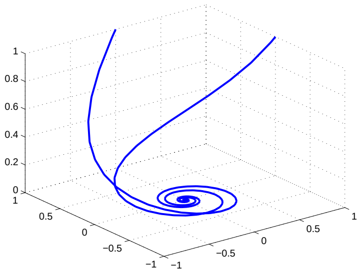
$K_d = 0.1$ $K_d = 0.02$



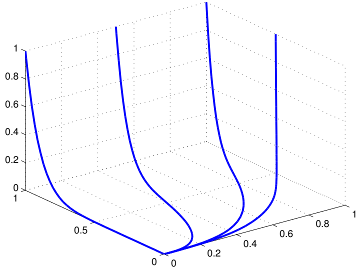
Two examples with three state variables

Examples of generalization to higher dimension

Stable "focus node"
(focus + one real eigenvalue)



Stable equilibrium with 3 distinct real eigenvalues
(generalization of 2 distinct real eigenvalues case)



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