Dynamics of opinion forming in structurally balanced social networks

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Abstract— **The aim of this paper is to shed light on how the social relationships between individuals influence their opinions in the case of structurally balanced social networks. If we represent a social network as a signed graph in which individuals are the nodes and the signs of the edges represent friendly or hostile relationships, then the property of structural balance corresponds to the social community being splittable into two antagonistic factions, each containing only friends. A classical example of this situation is a two-party political system. The paper studies the process of opinion forming on such a social community, starting from the observation that the property of structural balance is formally analogous to the monotonicity property of dynamical systems. The paper shows that under the assumption that individuals are positively influenced by their friends and negatively influenced by their enemies, monotone dynamical systems, due to their order-preserving solutions, are natural candidates to describe the highly predictable process of opinion forming on structurally balanced networks.**

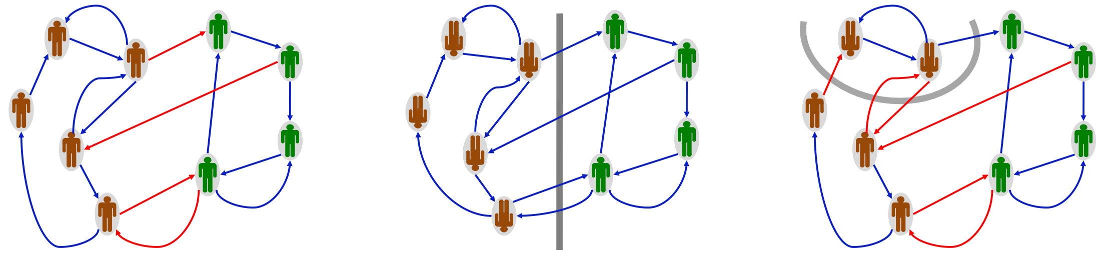
INTRODUCTION

In social network theory, a community of individuals characterized by friendly/hostile relationships is usually modeled as a signed graph having the individuals as nodes and their pairwise relationships as edges: an edge of positive weight expresses friendship, one of negative weight adversion or hostility [27], [7]. According to Heider theory of structural balance (or social balance) [13], in a balanced community structural tensions and conflicts are absent. This corresponds to the fact that the role of friends and enemies, determined locally by the bipartite relationships, is perfectly defined also on triads and, more generally, on a global scale. An equivalent characterization is in fact that the network splits into two factions such that each faction contains only friendly relationships while individuals belonging to different factions are linked only by antagonistic relationships [27]. In the graph-theoretical formulation of Cartwright and Harary [5], the lack of structural tensions corresponds to all cycles of the signed graph being positive, i.e., all having an even number of negative edges, see Fig. 1.

Following [5], in a perfectly balanced community it is reasonable to assume that for a person the point of view of a friend influences positively the process of forming an opinion about a subject; the opposite for an adversary. Quoting [5]: "the signed graph depicting the liking relations among a group of people will, then, also depict the potential influence structure of the group". Under this hypothesis, it is plausible to deduce that the outcome of an opinion forming process overlaps with the bipartition of the network: opinions are homogeneous within a faction and opposite with respect to those of the other faction. In this paper we ask ourself what kind of dynamics is suitable to represent this process of forming an opinion in a structurally balanced world of friends and adversaries.

In terms of dynamical systems, we can think of "influence" in the sense mentioned above as a directional derivative in opinion space, and of the Jacobian matrix of partial derivatives as the collection of all these influences. The principle stated above that the influences among the members of the community are depicted by their social relationships corresponds to identifying the signs of the entries of the Jacobian with those of the "sociomatrix" i.e., of the adjacency matrix of the signed graph describing the social network. The role played by friends and adversaries is assumed to be free from ambiguities, and this corresponds to constant sign of the partial derivatives in the entire opinion space. In dynamical systems theory, the systems whose Jacobians are sign constant at all points and such that the associated signed graph consists only of positive cycles form an important class of systems, called monotone systems [23], [22], [24]. Monotone systems are well-known for their dynamical properties: they respond in a predictable fashion to perturbations, as their solutions are "ordered" in the sense that they do not admit neither stable periodic orbits nor chaotic behavior [24]. Owing to their order-preserving flows, in many aspects monotone systems behave like 1-dimensional systems. Such notions of order are very appropriate for structurally balanced social networks, for which the pattern of opinions is completely predictable from the signed graphs depicting the social relationships [5]. Scope of this paper is to make the link between structural balance theory and monotone dynamical systems theory clear and formally precise.

A classical example where structural balance theory applies is two-party (or two-coalition) political systems [26]. In these systems we too often see that opinions within a faction are monolitic and antipodal to those of the other faction, and that discussion among the two factions is a wall-against-wall fight. Other cases in which structural balance has been suggested to correctly reproduce the phenomenology of a social community are for example duopolistic markets, rival business cartels [3], various case studies from anthropology [12] and social psychology [2], [21]. See [27], [7] for a more complete list of examples. In other contexts, notably in biological networks [6], [15], [14] and in on-line social networks [18], [17], [25], structural balance is not exact. One



A B C

Fig. 1. **Structurally balanced community.** (A): The community split into two factions such that members of the same faction are connected by friendly relationships (blue edges) and positively influence each other, while members of opposite factions are linked by adversary relationships (red edges) and negatively influence each other. All cycles and semicycles contain an even number of negative edges. (B): the gauge transformation, i.e., the switch of sign to all edges of the cut set (gray line), renders the signed graph completely blue. It corresponds to all individuals on one side of the cut set changing their mind simultaneously on their relationships with the other faction (in the drawing individuals are "flipped" for analogy with spins in Statistical Physics). (C): In this gauge transformation only the two individuals above the gray cut set switch side. The graph clearly remains bipartite. The three signed graphs in (A), (B) and (C) all are exactly structurally balanced.

can then try to quantify this amount of unbalance [18], [6], [15], [9], or study dynamical evolutions of the edge signs that lead to structural balance [1], [19]. These types of dynamics are fundamentally different from those investigated here, as our sociomatrices are and remain structurally balanced for all times.

If a major feature of a structurally balanced world is that the members of a community are influenced in their decision by the social network they form, a series of other properties of these systems admit interpretations in terms of monotone dynamics. One such property is that a small germ of opinion seems to be propagating unavoidably to the whole network if the network is connected. In structurally balanced systems, this often seems to be happening only due to the process of decision forming itself, regardless of the intrinsic value of the opinion (think of some decisions in the aforementioned two-party political systems). Monotone systems, thanks to their order-preserving solutions, also exhibit this behavior. We will show how for these systems the individual who seeds an idea first has a strong competitive advantage over both friends and rivals.

The signed graphs used in social network theory can be either undirected or directed [27], [7]. In the present context, an undirected edge corresponds to a mutual relationship (and mutual influence) between the two individuals connected by the edge, while a directed edge corresponds to an influence which is not reciprocated. In many instances of social networks, in fact, not all individuals have the same power of persuasion over their peers. In particular, the fact that an "opinion leader" may be influential for the opinions of its neighbors on the network (both friends and adversaries), does not mean that the implication has to reciprocate. Both the concepts of structural balance and of monotonicity extend to directed graphs in a similar manner. Also the graphical tests available in the literature coincide [27], [24].

If influences are associated with edges of the social network, it means that an individual with zero in-degree is unaffected by the opinion of the community (one with zero out-degree is instead unable to influence the community). At the other extreme, highly connected individuals are those influencing (or being influenced) the most. In particular, strong connectivity of a network means that all individuals have some influence power and are at the same time influenced by the community. A monotone dynamical system on a strongly connected graph is called strongly monotone [23]. The main characteristic of strongly monotone systems is that the order in the solutions is strict. This corresponds to the property that all individual in a strongly connected structurally balanced graph must necessarily take side: neutral opinions are not possible on such social networks.

Although the strength of the opinions at steady state depends on the precise functional form chosen for the dynamics, we already mentioned that in general the individuals with the highest in-degree achieve the strongest opinions. In our models this is true regardless of whether their relationships are friendly or hostile. We interpret this property by observing that both monotonicity of a system and structural balance of a social network are invariant to a particular class of operations which, for analogy with Ising spin glasses in Statistical Physics [4], we call gauge transformations. Consider the signed graph representing the social network and a cut set that splits the graph into two disconnected subgraphs. A change of sign on all edges intersecting the cut set cannot alter the signature of the cycles of the network (cut sets intersect cycles in an even number of edges). Such operations are called switching equivalences in the signed graph literature [28], or gauge transformations in the spin glass literature [4]. If we think of a signed graph as a spin glass, then a structurally balanced graph corresponds to a so-called Mattis model [20], in which the "disorder" introduced by the negative edges is only apparent, and can be completely eliminated by a suitable gauge transformation, see Fig. 1 (see [11] for an earlier formulation of a structurally balanced social network as a Mattis system). When applied to a monotone dynamical system, this transformation renders all entries of the Jacobian nonnegative, property known as Kamke condition in the literature [22]. Therefore the process of opinion forming of a two-party structurally balanced social

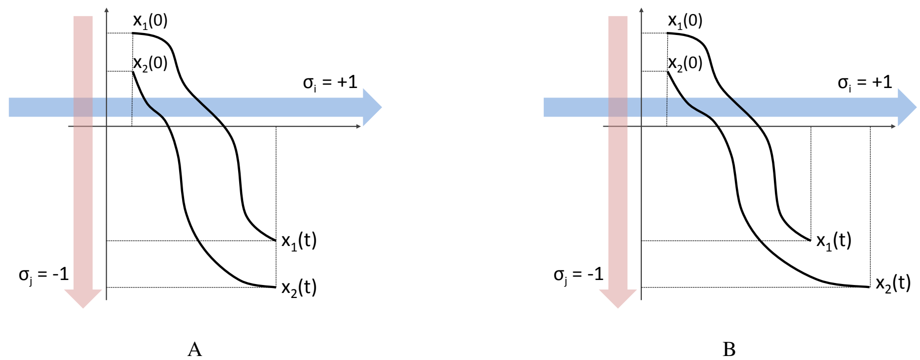


Fig. 2. **Examples of monotone and strongly monotone trajectories.** Given σ, a system like (1) is monotone (panel (A)) if initial conditions x_1, x_2 which respect the partial order σ (meaning $x_{1,i}(0) \leqslant x_{2,i}(0)$ when $\sigma_i = +1$, $x_{1,i}(0) \geqslant x_{2,i}(0)$ when $\sigma_i = -1$) induce solutions in (1) which respect the partial order σ for all times ($x_{1,i}(t) \leqslant x_{2,i}(t)$ when $\sigma_i = +1$, $x_{1,i}(t) \geqslant x_{2,i}(t)$ when $\sigma_i = -1$). It is strongly monotone (panel (B)) if initial conditions respecting the partial order σ and such that they differ in at least a coordinate ($x_{1,i}(0) \leqslant x_{2,i}(0)$ when $\sigma_i = +1$, $x_{1,i}(0) \geqslant x_{2,i}(0)$ when $\sigma_i = -1$, plus $x_{1,i}(0) \neq x_{2,i}(0)$ for some i) induce solutions in (1) which respect the partial order σ with strict inequality for all $t > 0$ along all coordinates ($x_{1,i}(t) < x_{2,i}(t)$ when $\sigma_i = +1$, $x_{1,i}(t) > x_{2,i}(t)$ when $\sigma_i = -1$).

network is always (dynamically) identical, up to the sign of the opinions, to that of a community with the same topology, but composed only of friends.

I. A DYNAMICAL MODEL FOR INFLUENCES.

Consider the dynamical system

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of opinions of the n individuals and the functions $f(\cdot)$ describe the process of opinion forming of the community. Assume $x = 0$ is a fixed point of (1). This is equivalent to assume that no opinion is formed unless at least one of the individuals has already an opinion at $t = 0$, i.e., unless $x(0) \neq 0$ in (1).

We model the influence of the j-th individual over the i-th individual by the partial derivative

$$F_{ij}(x) = \frac{\partial f_i(x)}{\partial x_j},$$

so that the matrix of pairwise influences

$$F(x) = \left[\tfrac{\partial f_i(x)}{\partial x_j}\right]_{i,j=1,\dots,n}$$

is the formal Jacobian of the system (1). We expect then that the influence of a friend is positive, $\frac{\partial f_i(x)}{\partial x_j} > 0$, and that of an adversary negative, $\frac{\partial f_i(x)}{\partial x_j} < 0$. We also expect that qualitatively these influences do not change sign if we compute them in different points x_1 and x_2 in opinions space. In formulas:

$$\mathrm{sign}(F(x_1)) = \mathrm{sign}(F(x_2)) \qquad \forall x_1,\, x_2 \in \mathbb{R}^n. \tag{2}$$

The "sign stability" condition (2) implies that if we define

$$A = \mathrm{sign}(F(x)), \tag{3}$$

then A is sign constant over the entire opinion space $\mathbb{R}^n$.

Notice that our considerations are more general than just taking the Jacobian linearization of (1) around an equilibrium point. In particular, the system (1) may have multiple equilibrium points, even with different stability characters. This is irrelevant to our discussion. Even the precise functional form of the $f(x)$ is not assumed to be known a priori, as long as it obeys (2). Furthermore, we do not consider our own current opinion as useful to reinforce it or to change our mind. On the contrary, we will normally consider $\frac{\partial f_i(x)}{\partial x_i} < 0$, i.e., opinions are gradually forgotten over time.

The following two different situations can be considered:

1) influences are always reciprocal:

$$F_{ij}(x) \neq 0 \quad \Longleftrightarrow \quad F_{ji}(x) \neq 0; \tag{4}$$

2) influences can be asymmetric

$$F_{ij}(x) \neq 0 \quad \not\Longleftrightarrow \quad F_{ji}(x) \neq 0. \tag{5}$$

We assume henceforth that $F_{ij}(x)$ and $F_{ji}(x)$ never have opposite signs:

$$F_{ij}(x)F_{ji}(x) \geqslant 0, \tag{6}$$

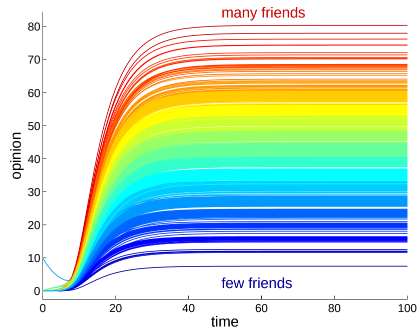
condition which is called sign symmetry in [22] and which corresponds to two individuals never perceiving reciprocal influences of opposite signs. From (3), conditions analogous to (4)-(5) hold for A: in the first case A is symmetric; in the second it need not be. The condition (6) instead becomes:

$$A_{ij}A_{ji} \geqslant 0. \tag{7}$$

A. The associated signed social community and its structural balance.

Under the sign stability condition (2), the *sociomatrix* of the signed social network can be identified with the matrix A. Associating social relationships with influences, as assumed here, means that the i-th individual considers the j-th individual a friend when $A_{ij} > 0$, an adversary when $A_{ij} < 0$, while when $A_{ij} = 0$ no relationship is perceived by the i-th individual. Therefore in this work the matrix A plays the double role of signature of the Jacobian of the influences and of sociomatrix of the signed social network.

For a symmetric A, assuming that the social community is structurally balanced means that all cycles in the (undirected)



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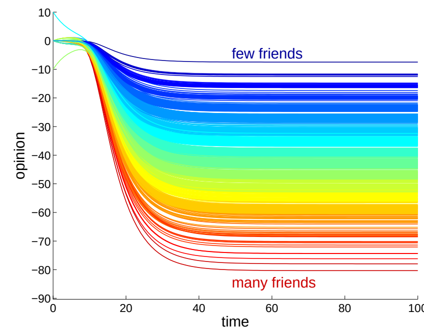

B

Fig. 3. **Collective opinions triggered by the opinion of one or two individuals, in the cooperative behavior case.** The color of a curve is proportional to the in-degree of the individual. Individuals with the highest in-degree form the strongest opinions. In panel (A) a single $x_i(0) > 0$ steers the whole community to a positive opinion; in panel (B) two individuals have contrasting initial conditions. When all influences are equal, the whole community is steered towards the opinion of the most connected non-zero opinioner.

graph of adjacency matrix A have to have positive sign [5]. When instead influences can be asymmetric, A is the adjacency matrix of a digraph. In social network theory, the notion of structural balance is extended to digraphs by looking at "semipaths" and "semicycles", i.e., undirected paths and undirected cycles of the underlying undirected signed graph obtained ignoring the direction of the edges [5], see Fig. 1(A) for an example. A necessary and sufficient condition for a digraph to admit an underlying undirected graph is (7), i.e., no negative directed cycle of length 2 exist in the signed digraph. Under this assumption, no cancellation appears when we take the "mirror" of A (i.e., A^T) and consider $A_u = \mathrm{sign}(A + A^T)$ as adjacency matrix of the underlying undirected graph. When this is possible, then a directed signed network is structurally balanced if and only if all undirected cycles of A_u have positive sign [5].

A sociomatrix A is reducible if there exists a permutation matrix P such that $PAP = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$ with A_1, A_3 square submatrices. A is irreducible otherwise. In terms of the graph of A, irreducibility corresponds to a strongly connected graph, i.e., a graph for which there exists a directed path between any pair of nodes. Irreducibility of A implies therefore that each individual is directly or indirectly influenced by the opinion of any of the other members of the community.

B. Monotone dynamical systems.

For a thorough introduction to the theory of monotone systems, the reader is referred to [23], [22], [24]. In $\mathbb{R}^n$, consider one of the orthants: $K_\sigma = \{x \in \mathbb{R}^n$ such that $Dx \geqslant 0\}$ where D is a diagonal matrix $D = \mathrm{diag}(\sigma)$ of diagonal elements $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \{\pm 1\}$, and denote by $x(t)$ the solution of (1) at time t in correspondence of the initial condition $x(0)$. The vector σ identifies a partial order for the n axes of $\mathbb{R}^n$, which can be the "natural" one when $\sigma_i = +1$, or the opposite when $\sigma_i = -1$.

The partial order generated by σ is normally indicated by the symbol $\leqslant_\sigma$: $x_1 \leqslant_\sigma x_2 \Longleftrightarrow x_2 - x_1 \in K_\sigma$. The system (1) is said *monotone* with respect to the partial order σ if for all initial conditions $x_1(0)$, $x_2(0)$ such that $x_1(0) \leqslant_\sigma x_2(0)$ one has $x_1(t) \leqslant_\sigma x_2(t)\ \forall\, t \geqslant 0$. Strict ordering is denoted $x_1 <_\sigma x_2$ and corresponds to $x_1 \leqslant_\sigma x_2$, $x_1 \neq x_2$, meaning that strict inequality must hold for at least one of the coordinates of x_1, x_2, but not necessarily for all. When inequality must hold for all coordinates of x_1, x_2 then we use the notation $\ll_\sigma$. The system (1) is said *strongly monotone* with respect to the partial order σ if for all initial conditions $x_1(0)$, $x_2(0)$ such that $x_1(0) <_\sigma x_2(0)$ one has $x_1(t) \ll_\sigma x_2(t)\ \forall\, t > 0$. See Fig. 2 for a graphical description of these definitions.

Monotonicity of a system can be verified in terms of the Jacobian matrix $F(x)$, via the so-called Kamke condition ([22], Lemma 2.1), which says that the system (1) is monotone with respect to the order σ if and only if

$$\sigma_i \sigma_j F_{ij}(x) \geqslant 0 \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, i, j = 1, \ldots, n \quad i \neq j. \tag{8}$$

From (2)-(3), it follows that the condition (8) can be stated equivalently in terms of A as

$$\sigma_i \sigma_j A_{ij} \geqslant 0 \qquad \forall\, i, j = 1, \ldots, n \quad i \neq j. \tag{9}$$

The condition (9) admits a graph-theoretical reformulation which is identical to that for structural balance (see e.g. [22]). The system (1) is monotone with respect to some orthant order if and only if all semicycles of length > 1 of the signed digraph of the sociomatrix A have positive sign. Therefore, under the assumption that our opinion is positively influenced by our friends and negatively by our adversaries, we can conclude that the dynamics of opinion forming in structurally balanced communities have indeed to obey a monotone dynamics.

Under the assumption (7), the condition (9) (and, similarly, (8)) covers both cases of symmetric and asymmetric influences. In fact, the non-strict inequality in (9) accounts exactly for situations in which $A_{ij} \neq 0$ while $A_{ji} = 0$, encountered in directed graphs.

If in addition to being monotone the sociomatrix A is also irreducible, then the system (1) is also strongly monotone [23]. Strong monotonicity implies that opinions are strictly ordered for all individuals. In terms of our social community, this irreducibility corresponds to the fact that all individuals have some influence power over the community, even the less influential members, and strict ordering translates into the fact that no individual can remain neutral to the influences of the community. Hence, whenever an opinion is seeded all individuals have to eventually take side. Following [24], a graph-theoretical test of strong monotonicity is that all directed cycles of the (strongly connected) digraph of A have to have positive sign.

II. Results

A. *Cooperative behavior: an all-friends world.*

A particular (trivial) case of structural balance is given by A with all non-negative entries. All individuals are friends and no tension ever emerges in decision making, except perhaps for a transient evolution (due to conflicting initial conditions). The corresponding system (1) is called *cooperative* in this case [22].

We analyze the following situations for the initial conditions

- a single individual has an opinion at $t = 0$;
- two different individuals have opposite opinions at $t = 0$.

From the definition of monotonicity, it follows that any initial condition $x(0) \geqslant 0$, $x_i(0) \neq 0$ for at least one i, implies that $x(t) \geqslant 0 \; \forall t > 0$. In particular, under the strong connectivity assumption, $x(t) > 0 \; \forall \, t > 0$, meaning that the opinion of the whole community gets influenced even by a single $x_i(0) \neq 0$. This situation is shown in Fig. 3(A) for the functional form $f(x)$ described in the Appendix. It can be observed that the strongest opinions are achieved by the most connected individuals (red lines mean high in-degree). In a similar way, $x(0) \leqslant 0$ implies $x(t) \leqslant 0 \; \forall \, t > 0$ (or $x(t) < 0 \; \forall \, t > 0$ when strongly connected).

The only case in which contrast can arise in a cooperative system is when two individuals have opposite opinions at $t = 0$. Such a contrast is not tolerated by a cooperative system, and in fact the whole community is steered to an unanimous opinion after a transient, see Fig. 3(B) for an example. Assume the i-th and the j-th nodes have opposite nonzero initial opinions, e.g. $x_i(0) > 0$ and $x_j(0) < 0$. Which of these opposite initial opinions will prevail depends on the strengths of $x_i(0)$ and $x_j(0)$, on the form of the $f(x)$ and on the connectivity of the i-th and j-th individuals.

B. *Two-party behavior and gauge transformations.*

A well-known property of a structurally balanced signed social network is that it can be partitioned into two disjoint antagonistic subcommunities. Each community contains only friends, while any two (related) individuals from different communities are adversaries. This means only $+1$ edges of A link members of the same party, while only -1 edges link members of different parties, see Fig. 1. From the sign stability condition (2), the same is true replacing A with the formal Jacobian $F(x)$. Consider the change of coordinates $y = Dx$, $D = \mathrm{diag}(\sigma)$ and σ a partial ordering of $\mathbb{R}^n$. Since $|y| = |x|$ and $F(x)$ is sign constant for all $x \in \mathbb{R}^n$, it follows that $\mathrm{sign}(F(x)) = \mathrm{sign}(F(y))$. From $D^{-1} = D$, the change of variable $y = Dx$ yields the new Jacobian $DF(Dx)D$. For analogy with the theory of Ising spin glasses [4], operations like

$$F(x) \to DF(y)D \tag{10}$$

are here called *gauge transformations*, and correspond to rearranging of the order of the n axes of $\mathbb{R}^n$ which modify the sign of the entries of the Jacobian, without altering its absolute values. In terms of the graph of A, a gauge transformation $A \to DAD$ corresponds to changing sign to all edges adjacent to the nodes corresponding to the -1 entries of σ. As directed cycles and semicycles share two (or zero) edges with each node, gauge transformations do not alter the signature of the cycles of the network. This is well-known in the Ising spin glass literature, see e.g. [10] (the extension to digraphs is completely straightforward). One says that operations like (10) can alter the "apparent disorder", while the "true disorder" (or "frustration") of the system is an invariant of (10). In particular, when A is structurally balanced the true disorder is zero. In Statistical Physics this case is called Mattis spin model [4]: an Ising model in "disguise" (the disguise being a gauge transformation). The Kamke condition rephrases this property in terms of $F(x)$. In fact, (8) implies that there exists a special ordering $\tilde{\sigma}$ for which the gauge transformed system $\tilde{F}(y) = \tilde{D}F(\tilde{D}x)\tilde{D}$, $\tilde{D} = \mathrm{diag}(\tilde{\sigma})$, is such that $\tilde{F}_{ij}(y) \geqslant 0 \; \forall \, i, j$, like in a cooperative system. In terms of the dynamics (1), this means that in a structurally balanced network the presence of adversaries does not alter the monotonic character of the opinion forming process: the dynamics is monotone regardless of the amount of apparent disorder present in the system. The only difference in the integral curves of (1) with respect to the cooperative case is that now the two parties converge (equally orderly) to opposite decisions, according to the faction to which each individual belongs to.

The case of opinions triggered by a single individual is shown in Fig. 4. In particular in this model (with the assumption of all identical kinetics adopted in the simulations, see Appendix), the strength of the opinion of an individual is not a function of the in-degree of friends or of adversaries alone, but only of the total in-degree of relationships of an individual regardless of their sign, see Fig. 4(A) *vs* (B).

III. Conclusion

While the connection between structural balance and monotonicity is not new [24], the novelty of this paper is the use of this connection to draw conclusions on plausible opinion dynamics taking place on structurally balanced communities. A structurally balanced network represents a perfectly polarized community in which the drawing of a line separating friends from enemies is always an unambiguous process. It is this lack of ambiguity that yields the high predictability of opinions. The key assumption for this to

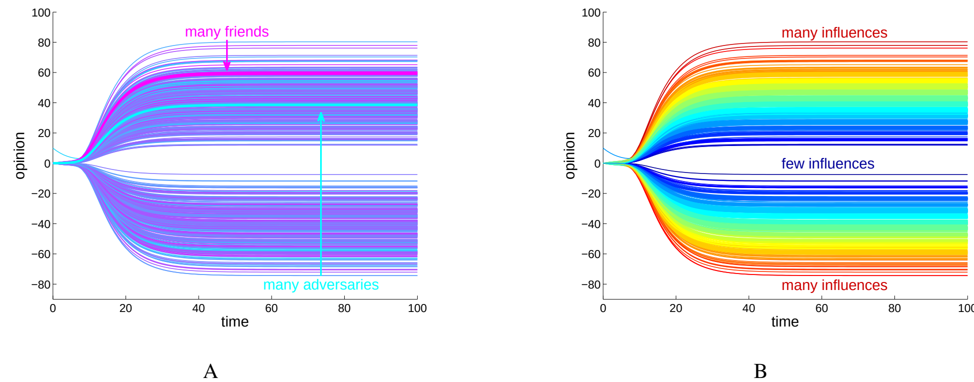

Fig. 4. **Collective opinions triggered by the opinion of a single individual, in the structurally balanced case.** Both (A) and (B) show how the community becomes polarized into two factions with opposite opinions. (A): The color of a curve is proportional to the difference in the in-degree between friends and adversaries. The two highlighted curves represent the individuals with the most of friends (pink) and adversaries (cyan). (B): for the same dynamics as in (A) the color now represents the total in-degree of an individual, regardless of the sign of the relationship. Clearly the strength of an opinion depends on the total number of relationships, rather than on the proportion friends/adversaries.

happen, namely that the opinions of friends exercise a positive influence and those of enemies a negative one, is realistic in this context. Most importantly, this assumption is needed only in *qualitative* terms, in the sense that it is not related to the specific values assumed by the F_{ij} but only to their sign. This is important in our case, as the precise functional form of a dynamical process of opinion forming is necessarily known only in qualitative terms.

APPENDIX

A more detailed model formulation: decentralized additive nonlinear systems

Any sufficiently regular $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ whose formal Jacobian $F(x)$ obeys to the Kamke condition is acceptable for the purposes of this study. In this Appendix we introduce a special form for the ODEs (1) of the paper, which is then used to simulate monotone dynamical systems.

For the i-th component of the vector field $f(x)$ in (1) consider the following ODE:

$$\dot{x}_i = f_i(x) = \sum_{j=1}^{n} A_{ij}\psi_{ij}(x_j) - \lambda_i x_i, \tag{11}$$

$i = 1, \ldots, n$, where

- $\psi_{ij} = \psi_{ij}(x_j)$ is Lipschitz continuous, depends only on the j-th state and is such that $\frac{\partial \psi_{ij}}{\partial x_j} \geqslant 0$, $\forall\, x_j \in \mathbb{R}$;
- $\lambda_i > 0$, i.e., an exponential decay affects the opinions of all individuals;
- $A_{ij} \in \{0, \pm 1\}$ and $A = (A_{ij})$ structurally balanced, that is, $\exists\ \tilde{D} = \mathrm{diag}(\tilde{\sigma})$ such that $\tilde{D}A\tilde{D}$ has all nonnegative entries.

The form (11) corresponds to an opinion forming process in which at each node the influences of the neighbors is additive, weighted by the signed sociomatrix A, and damped by the forgetting factor λ_i. The functionals $\psi_{ij} : \mathbb{R} \to \mathbb{R}$ are local (i.e., only the state x_j of the j-th individual matters) and nondecreasing: the higher is x_j the higher $\psi_{ij}(x_j)$. The monotonicity of the functionals reflects the fact that the opinion expressed by the j-th individual (i.e., $\psi_{ij}(x_j)$) in his/her relationships is coherent with his/her "true" opinion (i.e., x_j). The sociomatrix A specifies if the opinion expressed by the j-th individual ($\psi_{ij}(x_j)$) influences positively or negatively the i-th neighbor (i.e., if $A_{ij}\psi_{ij}(x_j)$ is positive or negative). The degradation term λ_i has the form of an exponential decay, and models a forgetting factor in the opinion of each individual. Notice that since structural balance and monotonicity are checked on the off-diagonal part of the Jacobian, the presence of λ_i does not interfere with these properties of the system. It is however necessary to have the solutions of (14) not diverging to $\pm\infty$. Eq. (11) can therefore be written as

$$f_i(x) = \sum_{j \in friends(i)} \psi_{ij}(x_j) - \sum_{j \in advers(i)} \psi_{ij}(x_j) - \lambda_i x_i. \tag{12}$$

A special case of (12) is the following

$$\psi_{ij}(x_j) = \psi_j(x_j) \qquad \forall\ i = 1, \ldots, n$$

corresponding to a node influencing all its neighbors in the same manner (up to the sign, which is contained in A). In this case, (1) of the paper can be written more compactly as

$$\dot{x} = A\psi(x) - \Lambda x \tag{13}$$

where $\psi(x) = \begin{bmatrix} \psi_1(x_1) & \ldots & \psi_n(x_n) \end{bmatrix}^T$ and Λ is a diagonal matrix $\Lambda = \mathrm{diag}(\begin{bmatrix} \lambda_1 & \ldots & \lambda_n \end{bmatrix})$. Nonlinear systems with a structure like (13) are sometimes called Persidskii systems [16].

In this study we have chosen to work with monotonic functionals derived by the so-called Michaelis-Menten kinetic forms, widely used in mathematical biology to describe reaction rates [8]. These functions are the following:

$$\psi_j(x_j) = \frac{x_j}{\theta_j + |x_j|}, \qquad x_j \in \mathbb{R} \tag{14}$$

where $\theta_j > 0$ is called the half-rate constant, and represents the value of x_j at which ψ_j reaches the value of $1/2$. The peculiarity of the ψ_j is in fact that the rate it describes saturates for large values of x_j:

$$lim_{x_j \to \pm\infty} \psi_j(x_j) = \pm 1,$$

i.e., opinions are not formed too quickly. The absolute value in the denominator of (14) is needed to have a well-defined expression also for negative x_j (in biological models x normally represents concentrations, hence the negative values are disregarded).

Since
$$\frac{\partial \psi_j(x_j)}{\partial x_j} = \frac{\theta_j}{(\theta_j + |x_j|)^2} > 0$$

from (12) one has

$$F_{ij}(x) = \frac{\partial f_i(x)}{\partial x_j} = \begin{cases} \frac{\theta_j}{(\theta_j+|x_j|)^2} & \text{if } j \in friends(i) \\ \frac{-\theta_j}{(\theta_j+|x_j|)^2} & \text{if } j \in advers(i) \end{cases} \quad (15)$$

i.e., $\text{sign}(F_{ij}(x))$ is always constant for all $x \in \mathbb{R}^n$, as required for monotone systems. In particular, then, a change of partial order σ on the axes, $y = Dx$, $D = \text{diag}(\sigma)$, does not alter the Jacobian $F(x)$. From (15), in fact, $F_{ij}(x) = F_{ij}(y)$. From the sign stability of $F(x)$, it follows that $\text{sign}(F_{ij}(x)x_j) = \text{sign}(A_{ij}\psi_{ij}(x_j))$, which is enough to guarantee that the Jacobian linearization faithfully reflects the behavior of the original nonlinear system.

In the simulations of the paper, the half-rates θ_j and the degradation rates λ_i are fixed all equal.

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