Short Papers

Redundant Robotic Chains on Riemannian Submersions

Claudio Altafini

Abstract—**The main scope of this paper is to introduce the notion of Riemannian submersion for the modeling and control of certain types of redundant robotic chains. In the robotics literature, the redundant case is normally treated only in numerical terms, as the need to resort to pseudoinversion techniques is usually considered a barrier to the use of analytic or geometric methods. Using Riemannian submersions, however, we can single out a particular type of inverse, the horizontal lift, with distinguished properties with respect to the infinitely many possible others. Quite remarkably, for a wide class of robotic chains, characterized by the vanishing of the curvature tensor, such horizontal lift coincides with the curve obtained from the Moore–Penrose pseudoinverse of the Jacobian.**

Index Terms—**Motion control, product of exponentials formula, pseudoinverse, redundant robotic chains, Riemannian submersions.**

I. Introduction

From a mathematical viewpoint, a robotic chain can be seen as a mechanical control system having as configuration space the manifold in which the joint/link variables are living, and control inputs that are the torques/forces applied at the same joint/link. See any of the many books on modeling and control of robotic manipulators, for example, [17] and [20]. By specifying the inertia property of each joint/link, a set of forced Euler–Lagrange equations can be obtained for the chain. Bedrossian and Spong showed in [6] the existence of a class of robotic chains having Riemannian curvature that is locally vanishing, once friction phenomena and potential energy are neglected. When the Riemannian curvature is vanishing, the Euler–Lagrange equations of the robotic chain can be linearized by means of an isometry (rather than by the usual feedback linearization method). In this case, the mechanical system is also said to be "flat" and its model space is an Euclidean cylinder.

The workspace, i.e., the space in which the end-effector lives, is $SE(3)$. Since the dimension of $SE(3)$ is six, if the robotic chain has more than six degrees of freedom (DOFs), then (in the generic case) the system is redundant. Only "flat" redundant robotic chains are treated hereafter. Obviously, this is one of the situations in which we can assume that the forward kinematic map (from joint space to workspace) is surjective, and we can use the same map to push the Euler–Lagrange equations from joint space to workspace, once we have chosen on it a suitable Riemannian structure, here the so-called double-geodesic metric (denoted by the subindex "dg") [8], [18] for rotations and translations. If the corresponding metric tensor is chosen to be the identity, $M_{\rm dg} = I$, then, out of the singularities, the forward kinematics becomes a Riemannian submersion between an Abelian group and the noncommutative group $SE(3)$. Furthermore, the forward kinematics can be shown to be the projection map of a locally trivial fiber bundle over $SE(3)$. Then, many well-known facts of redundant robotic chains can be given a geometric interpretation. For example, at each point, the space of internal motions (i.e., the joint movements not affecting the end-effector) is the fiber over the same point; the repeatability (or cyclicity) of motion [4], [19] corresponds to the integrability of the horizontal distribution of the submersion or, equivalently, to the lack of "geometric phase" on the fiber variables. Any Riemannian submersion gives a preferred "geometric" way to pull back vectors to the (larger) source manifold, called the horizontal lift. For the case at hand here, the pullback is from workspace to the joint space, and it is obviously the inverse kinematics. Due to the rectangular Jacobian, the inverse kinematics for the redundant case lacks a closed-form symbolic solution. Common practice in robotics is to resort to pseudoinversion and to rely only on numerical integration of the differential inverse kinematics, thus obtaining inverse maps that lack any form of invariance. In the Riemannian context, the horizontal lift is a length- (and energy-) preserving map and provides an intrinsic geometric notion of inverse kinematics, free from numerical schemes. The main result of this paper is *Theorem III.2*, which states that for Abelian joint spaces, the (unique) horizontal lift of vector fields coincides with the Moore–Penrose pseudoinverse commonly used in robotics [13].

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The author was with the Division of Optimization and Systems Theory, Department of Mathematics, Royal Institute of Technology, SE-10044 Stockholm, Sweden. He is now with the SISSA/ISAS International School for Advanced Studies, 34014 Trieste, Italy (e-mail: altafini@sissa.it).

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II. Robotic Chains and Riemannian Submersions

In order to emphasize the intrinsic geometric properties of the robotic chain, we use the formalism of the *product of exponentials* proposed as first by Brockett, see [8] and [17] for details. Each rigid body transformation generates a one-parameter subgroup of $SE(3)$ or a translation of it out of the identity. The *forward kinematics* of the open robotic chain gives the description of the motion of the end-effector in terms of the joints/links. It can be seen as the smooth map

$$\begin{aligned} \rho:\ &\mathbf{Q} \to SE(3) \\ \mathbf{q} = [\,\mathbf{q}_1 \quad \dots \quad \mathbf{q}_n\,] &\mapsto g = \rho(\mathbf{q}). \end{aligned} \tag{1}$$

The type of joints/link we consider in this paper are a special type of lower pair joints, characterized by having a single DOF each and a joint space $\mathbf{Q}$ which is an Abelian group. The movements of the end-effector can be represented as a product of exponentials of the single 1-DOF screw motions. The (left invariant version of the) product of exponentials formula

$$g(\mathbf{q}) = g(0)e^{V_1\mathbf{q}^1}\dots e^{V_n\mathbf{q}^n} \tag{2}$$

is obtained by expressing all the infinitesimal generators of the one-parameter subgroups in the same frame, in (2), the end-effector frame. Differentiating (1), using the natural parallelism of any Abelian group and left invariance of $TSE(3)$, we obtain the *differential forward kinematics*

$$\begin{aligned} \rho_*:\ &T_{\mathbf{q}}\mathbf{Q} \to \mathfrak{se}(3) \\ &\dot{\mathbf{q}} \mapsto X_{\rho(\mathbf{q})} = J^b(\mathbf{q})\dot{\mathbf{q}} \end{aligned} \tag{3}$$

where $\dot{\mathbf{q}}$ corresponds to the usual notion of velocity in coordinates $\dot{\mathbf{q}} = \dot{\mathbf{q}}^i(\partial/\partial\mathbf{q}^i)$ (summation convention).

The Jacobian of the product of exponentials ρ is $J^b(\mathbf{q}) = g^{-1}(\mathbf{q})(\partial g/\partial \mathbf{q})$, and has the explicit expression

$$J^b(\mathbf{q}) = \begin{bmatrix} \mathrm{Ad}^{-1}_{g(0)e^{V_1\mathbf{q}^1}\dots e^{V_n\mathbf{q}^n}}\xi_1 & \dots & \mathrm{Ad}^{-1}_{g(0)e^{V_n\mathbf{q}^n}}\xi_n \end{bmatrix} \quad (4)$$

where $V_i \in \mathfrak{se}(3)$.

A. Joint Space Dynamic Equations

In joint space $\mathbf{Q}$, the Lagrangian of the manipulator is $L(\mathbf{q},\dot{\mathbf{q}}) = T(\mathbf{q},\dot{\mathbf{q}}) - V(\mathbf{q})$, where the kinetic energy defines an inner product in $T\mathbf{Q}: T(\mathbf{q},\dot{\mathbf{q}}) = \langle \dot{\mathbf{q}}, \dot{\mathbf{q}}\rangle = (1/2)\dot{\mathbf{q}}^T M(\mathbf{q})\dot{\mathbf{q}}$, with $M(\mathbf{q})$ the manipulator-generalized inertia matrix. $M(\mathbf{q})$ is positive definite and symmetric, therefore, it constitutes a well-defined metric tensor for the joint space and gives to $\mathbf{Q}$ the structure of a Riemannian manifold. Calling τ the external generalized forces, the Lagrange equations admit the usual expression for a robotic chain

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathrm{d}V(\mathbf{q}) = \tau \quad (5)$$

where the Coriolis matrix can be expressed in terms of the Christoffel symbols associated with the metric tensor $M(\mathbf{q}): C_{ij}(\mathbf{q},\dot{\mathbf{q}}) = (1/2)M_{ki}\Gamma^k_{jl}\dot{\mathbf{q}}^l$. If $M^{ij} = \left(M^{-1}\right)_{ij}$ is used to raise the indexes, then by premultiplying with $M^{-1}(\mathbf{q})$ and using the Euclidean local coordinates $\mathbf{q}^1,\dots,\mathbf{q}^n$

$$\ddot{\mathbf{q}}^k + \Gamma^k_{ij}\dot{\mathbf{q}}^i\dot{\mathbf{q}}^j = M^{kj}\left(\tau_j - \mathrm{d}V_j\right). \quad (6)$$

The Γ^k_{ij}'s defines the affine connection ∇, so that (6) can be rewritten as

$$\nabla_{\dot{\mathbf{q}}}\dot{\mathbf{q}} = M^{-1}\left(\tau - \mathrm{d}V(\mathbf{q})\right). \quad (7)$$

In the following, we will neglect $V(\cdot)$. In coordinates, the torsion-free property of ∇ can be expressed in terms of symmetry in the Christoffel symbols $\Gamma^k_{ij} = \Gamma^k_{ji}$. Furthermore, by assumption, ∇ is also locally flat, i.e., the curvature tensor vanishes. An alternative characterization is obtained via isometric transformations [5]. Recall that an *isometry* φ is a bijective map between Riemannian manifolds that preserves the inner product

$$\langle X, Y\rangle_x = \langle \varphi_* X, \varphi_* Y\rangle_{\varphi(x)} \quad (8)$$

where X and Y are vector fields on the source manifold (of metric $\langle\cdot,\cdot\rangle_x$). For flat manifolds, there exists an isometry (see, for example, [16, Prop. 5.6])

$$\varphi: (\mathbf{Q}, M(\mathbf{q})) \to (\tilde{\mathbf{Q}}, I) \quad (9)$$

with connection $\tilde\nabla$ defined as the push-forward of ∇

$$\tilde\nabla_{\varphi_* X}(\varphi_* Y) = \varphi_*(\nabla_X Y) \quad (10)$$

and such that the $\tilde\Gamma^k_{ij}$ of $\tilde\nabla$ are all equally 0. Finding φ means normally solving the system of equations for the factorization of M, and then integrating them. This is obtained as follows. Since $M(\mathbf{q})$ is symmetric, we can write it as $M(\mathbf{q}) = N(\mathbf{q})^T N(\mathbf{q})$, with $N(\mathbf{q})$ the Jacobian of an isometry $\forall\, \mathbf{q}$. From the spectral theorem for symmetric matrices, all the eigenvalues of M, $\lambda_1,\dots,\lambda_n$ (counted with multiplicity) are real, and M admits the factorization

$$M = P^T \mathrm{diag}(\lambda_1,\dots,\lambda_n)P$$

where P is an orthogonal matrix having as rows the (normalized) eigenvectors. Furthermore, positive definiteness of M implies that $\lambda_i > 0$, therefore

$$N = \mathrm{diag}\left(\sqrt{\lambda_1},\dots\sqrt{\lambda_n}\right)P = D_1 P \quad P \in O(n).$$

Call $\tilde{\mathbf{q}}$ the new state, $\tilde{\mathbf{q}} = \varphi(\mathbf{q})$, such that differentiating $\dot{\tilde{\mathbf{q}}} = \varphi_*(\mathbf{q}) = N(\mathbf{q})\dot{\mathbf{q}}$. $N(\mathbf{q})$ is the Jacobian of a diffeomorphism, therefore, $N^{-1}(\varphi^{-1}(\tilde{\mathbf{q}}))$ is well defined and nonsingular. The isometry is easily verified

$$\begin{aligned}\langle \dot{\tilde{\mathbf{q}}}, \dot{\tilde{\mathbf{q}}}\rangle_{\tilde{\mathbf{Q}}} &= \varphi_*^T(\mathbf{q})\varphi_*(\mathbf{q}) \\ &= \dot{\mathbf{q}}^T N^T(\mathbf{q})N(\mathbf{q})\dot{\mathbf{q}} = \langle \dot{\mathbf{q}}, \dot{\mathbf{q}}\rangle_{\mathbf{Q}}. \end{aligned} \quad (11)$$

Without entering into the details (see also [5] and [14] for a thorough treatment), in the velocity phase space $T_{\mathbf{q}}\mathbf{Q}$ this can be expressed as follows. If $\mathbf{q}$ and v are configuration and velocity coordinates in $T_{\mathbf{q}}\mathbf{Q}$ and $\tilde{\mathbf{q}}$, $\tilde v$ in $T_{\tilde{\mathbf{q}}}\tilde{\mathbf{Q}}$, then the full transformation is

$$\{\tilde{\mathbf{q}}, \tilde v, \tilde\tau\} = \left\{\varphi(\mathbf{q}), Nv, N^{-T}\tau\right\}.$$

Notice that this is different from the usual computed torque method, which consists of feedback linearizing the system by means of $\tau = C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + M(\mathbf{q})\tilde\tau$. The model space $(\mathbf{Q}, M(\mathbf{q}))$ is the n-dimensional Euclidean cylinder, and the isometry φ linearizes the Euler–Lagrange equations

$$\tilde\nabla_{\dot{\tilde{\mathbf{q}}}}\dot{\tilde{\mathbf{q}}} = \ddot{\tilde{\mathbf{q}}} = \tilde\tau. \quad (12)$$

B. Forward Kinematics as a Riemannian Submersion

We assume in the following to consider only the points $\mathbf{q} \in \mathbf{Q}$ in which the smooth map ρ is locally surjective in $SE(3)$, i.e., ρ_* has full row rank.

A1 *Assume that at* $\mathbf{q} \in \mathbf{Q}$ rank $(\rho_*(\mathbf{q})) = 6$.

Then, by the open mapping theorem, ρ is locally surjective on a full neighborhood of $\rho(\mathbf{q})$. When assumption **A1** is verified, we say that $\rho: \mathbf{Q} \to SE(3)$ is a locally surjective *submersion*. Necessary condition is that the dimension of $\mathbf{Q}$ is at least six. For any $g \in SE(3)$ the *fiber* over g, $\rho^{-1}(g)$, is a closed embedded submanifold of $\mathbf{Q}$ of dimension $n-6$ by the implicit function theorem. In robotics, $\rho^{-1}(g)$ is normally called the *space of internal motions*, i.e., the set of joint movements that do not affect the end-effector. Since $\mathbf{Q}$ is a Riemannian manifold, at each $\mathbf{q} \in \mathbf{Q}$, $T_{\mathbf{q}}\mathbf{Q}$ can be decomposed into an orthogonal direct sum

$$T_q\mathbf{Q} = \mathcal{H}_q \oplus \mathcal{V}_q \quad (13)$$

where $\mathcal{V}_{\mathbf{q}}$ is the tangent space to the fiber

$$\mathcal{V}_{\mathbf{q}} = \ker \rho_*|_{\mathbf{q}} = T_{\mathbf{q}}\rho^{-1}\left(\rho(\mathbf{q})\right)$$

and $\mathcal{H}_{\mathbf{q}} = \mathcal{V}_{\mathbf{q}}^\perp$ is the horizontal space. The submersion is said to be a *local Riemannian submersion* if it preserves the length of the horizontal vectors

$$\langle \dot{\mathbf{q}}^h_X, \dot{\mathbf{q}}^h_Y\rangle_{\mathbf{Q}} = \langle \rho_*\dot{\mathbf{q}}^h_X, \rho_*\dot{\mathbf{q}}^h_Y\rangle_{SE(3)} \qquad \forall\, \dot{\mathbf{q}}^h_X, \dot{\mathbf{q}}^h_Y \in \mathcal{H}_{\mathbf{q}}.$$

In other words, a Riemannian submersion is such that, for any $X \in \mathfrak{se}(3)$, there is a unique $\dot{\mathbf{q}}^h_X \in T_{\mathbf{q}}Q$ that is (faithfully) ρ-related to $X: \rho_*\dot{\mathbf{q}}^h_X = X_{\rho(\mathbf{q})}\ \forall\, \mathbf{q} \in \mathbf{Q}$. $\dot{\mathbf{q}}^h_X$ is called the *horizontal lift* of X at $\mathbf{q}$. From (13), then, each vector $\dot{\mathbf{q}}$ of $T_{\mathbf{q}}\mathbf{Q}$ admits the decomposition $\dot{\mathbf{q}} = \dot{\mathbf{q}}^h_X + \dot{\mathbf{q}}^v_X$, where $\dot{\mathbf{q}}^h_X \in \mathcal{H}_{\mathbf{q}}$ and $\dot{\mathbf{q}}^v_X \in \mathcal{V}_{\mathbf{q}}$.

Assumption **A1** alone does not guarantee that a submersion is Riemannian, because of the presence of singularities in the robotic chain. We therefore need to consider the stronger condition.

A2 *Assume that at* $\mathbf{q} \in \mathbf{Q}$ $\dim\mathcal{H}_{\mathbf{q}} = 6$ *and* $\dim\mathcal{V}_{\mathbf{q}} = n-6$.

The set of points in which **A2** is satisfied is open and dense in $\mathbf{Q}$. The second part of **A2** will be needed in order to consider isomorphic fibers. In robotics, the singularities that occur when $\dim\mathcal{V}_{\mathbf{q}} < n-6$ are called algorithmic singularities [3].

The forward kinematic map ρ takes a Euclidean space to a noncommutative group. In fact, while the joint space is flat, $SE(3)$ has a nonnull curvature. The following proposition says that it does so respecting the lengths of vectors, whenever the horizontal vectors do not degenerate.

Proposition II.1: Under the assumption **A2**, the forward kinematic map ρ is a Riemannian submersion.

Proof: The result is a consequence of the observation above that $SE(3)$ is the group of motions of $\mathbb{R}^3$, and (provided we choose the metric tensor $M_{\rm dg} = I$) that the linearization φ can reduce the metric of $\mathbf{Q}$ to the identity. Consider $\mathbb{R}^6$ endowed with the $\overset{\rm dg}{\nabla}$ connection instead of its Riemannian (Euclidean) connection. $\overset{\rm dg}{\nabla}$ is metric ([16, p. 112]) but not symmetric (the torsion tensor $T(X,Y) = \nabla_X Y - \nabla_Y X$ is nonnull, since $[X,Y] = 0$). Therefore, for horizontal vectors $\dot{\mathbf{q}}_X, \dot{\mathbf{q}}_Y \in \mathcal{H}_{\mathbf{q}}$

$$\langle \dot{\mathbf{q}}_X, \dot{\mathbf{q}}_Y \rangle_{\mathbf{Q}} = \langle \rho_*(\dot{\mathbf{q}}_X), \rho_*(\dot{\mathbf{q}}_Y) \rangle_{\mathbb{R}^6}$$

i.e., $\mathbf{Q} \rightarrow \left(\mathbb{R}^6, \overset{\rm dg}{\nabla}\right)$ is an isometric submersion. For the orthogonal subalgebra of $\mathfrak{se}(3) = \mathfrak{so}(3) \circledS \mathbb{R}^3$, we have the isomorphism $\mathfrak{so}(3) \simeq (\mathbb{R}^3, \times)$. The cross product induces a Lie algebra structure on $\mathbb{R}^3$, which is compatible with the Euclidean inner product. $x, y \in \mathbb{R}^3$ implies $\langle x, y \rangle_{\mathbb{R}^3} = \langle \widehat{x}, \widehat{y} \rangle_{\mathfrak{so}(3)}$. Endowing $\mathbb{R}^3$ with cross product and keeping $\overset{\rm dg}{\nabla}$ preserves lengths and makes $\overset{\rm dg}{\nabla}$ symmetric. Therefore, for $\dot{\mathbf{q}}_X, \dot{\mathbf{q}}_Y \in \mathcal{H}_{\mathbf{q}}$

$$\begin{aligned} \langle \dot{\mathbf{q}}_X, \dot{\mathbf{q}}_Y \rangle_{\mathbf{Q}} &= \langle \rho_*(\dot{\mathbf{q}}_X), \rho_*(\dot{\mathbf{q}}_Y) \rangle_{\mathfrak{se}(3)} \\ &= \dot{\mathbf{q}}_X^T {J^b}^T(\mathbf{q}_X) M_{\rm dg} J^b(\mathbf{q}_Y) \dot{\mathbf{q}}_Y. \end{aligned}$$

Such a property is preserved through φ as it is straightforward to verify from (11). ∎

Alternatively, one could prove the proposition above by considering the "absolute parallelism" of the Lie group, obtained by regarding it as a trivial reductive homogeneous space with respect to the left action on itself, i.e., endowing $SE(3)$ with the so-called $(-)$ connection (the flat connection with torsion tensor $T(X, Y) = -[X, Y]$, see [10]) and then transforming to $\overset{\rm dg}{\nabla}$. A nonnull curvature on a Riemannian manifold is a measure of the nonintegrability of the horizontal distribution. The crucial step of the whole reasoning here is that the curvature of $SE(3)$ can be canceled by a change of connection which does not modify the length of the vectors and the angles between them.

III. Horizontal Lift for a Robotic Chain

In the following, we call ρ^{*h} the pullback map, i.e., the linear map between tangent spaces $\mathfrak{se}(3) \rightarrow T_{\mathbf{q}}Q$ giving the horizontal lift of $X \in \mathfrak{se}(3)$ at each $\mathbf{q} \in \mathbf{Q}$

$$\rho^{*h}\left(X_{\rho(\mathbf{q})}\right) = \rho^{*h} X_{\rho(\mathbf{q})} = \dot{\mathbf{q}}_X^h. \tag{14}$$

In a Riemannian submersion, just like a vector field admits a horizontal lift, so does any curve in $SE(3)$. Given a path $\gamma(t) \in C^1(SE(3))$, the horizontal lift of γ is any path $c(t) \in \mathbf{Q}$, such that $\dot{c}(t)$ is horizontal for all t and $\rho(c(t)) = \gamma(t)$. In our case, the situation simplifies a lot, since $\mathbf{Q}$ is Abelian (and therefore, complete as a metric space). We can use the following theorem by Hermann [12]: call $\mathcal{H}$ and $\mathcal{V}$ the horizontal and vertical distributions (i.e., $\mathcal{H} = \bigcup_{\mathbf{q}} \mathcal{H}_{\mathbf{q}}$ and $\mathcal{V} = \bigcup_{\mathbf{q}} \mathcal{V}_{\mathbf{q}}$ for $\mathbf{q} \in \mathbf{Q}$ satisfying **A2**).

Theorem III.1: $(\mathbf{Q}, M(\mathbf{q}))$ complete and ρ a Riemannian submersion imply that $\mathcal{H}$ is complete and that ρ is the projection map of a locally trivial fiber bundle over $SE(3)$

$$\rho:\ \mathbf{Q} \rightarrow SE(3) \tag{15}$$

with fibers locally isomorphic to $\mathbb{R}^{n-6}$. Furthermore, the induced connection is complete.

Completeness here refers to the corresponding object being defined for all times. This property is important for our purposes because it implies the following fact.

Corollary III.1: For all paths $\gamma(t) \in SE(3)$ starting at γ_0 and any $\mathbf{q}_0 \in \rho^{-1}(\gamma_0)$, there exists a unique horizontal lift $c(t) \in \mathbf{Q}$ of $\gamma(t)$ starting at $\mathbf{q}_0$.

Notice that the local triviality statement of *Theorem III.1* is implicit in the definition of the fiber bundle and, due to assumption **A2**, none of our considerations can be given a global character. Equation (15) follows from the existence of a locally isometric diffeomorphism between $SE(3)$ and $\mathbb{R}^6$ (given, for example, by choosing Euler angles on $SO(3)$ and Cartesian coordinates on $\mathbb{R}^3$), i.e., the two horizontal maps of the diagram are local isometries

$$\begin{array}{ccc} \mathbb{R}^6 \times \mathbb{R}^{n-6} & \longleftrightarrow & \mathbf{Q} \\ \big\downarrow & & \big\downarrow \rho \\ \mathbb{R}^6 & \longleftrightarrow & SE(3). \end{array}$$

The isometry follows from *Proposition II.1*. The local triviality of the fiber bundle induces a direct product splitting also on the tangent bundle.

Corollary III.2: $\mathcal{H}$ and $\mathcal{V}$ are both locally integrable distributions.

While integrability of $\mathcal{V}$ is trivial, the integrability of $\mathcal{H}$ holds only because $\mathbf{Q}$ is Abelian. In this case, in fact, closed paths on $\mathcal{H}$ do not give any "geometric phase" on $\mathcal{V}$. In the literature, this phenomenon is normally called *repeatability* [19] or cyclicity of tracking [4], since it corresponds to the fact that applying a closed-loop trajectory $\gamma(t) \in SE(3)$ (contained in a simply connected open set that satisfies **A1**), the inverse kinematics produces closed loops that "do not drift." The integrability of $\mathcal{H}$ allows obtaining a relation similar to (10) between $\overset{\rm dg}{\nabla}$ and ∇.

Proposition III.1: Consider $(\mathbf{Q}, M(\mathbf{q}))$ and $(SE(3), M_{\rm dg})$

$$\nabla_{\dot{\mathbf{q}}_X^h} \dot{\mathbf{q}}_Y^h = \left(\nabla_{\dot{\mathbf{q}}_X} \dot{\mathbf{q}}_Y\right)^h = \rho^{*h}\left(\overset{\rm dg}{\nabla}_{X_{\rho(\mathbf{q})}} Y_{\rho(\mathbf{q})}\right) \tag{16}$$

where $\dot{\mathbf{q}}_X^h$ and $\dot{\mathbf{q}}_Y^h \in T_{\mathbf{q}}\mathbf{Q}$ are the horizontal lifts of $X_{\rho(\mathbf{q})}$ and $Y_{\rho(\mathbf{q})}$, and $\dot{\mathbf{q}}_X|_{\mathcal{H}_{\mathbf{q}}} = \dot{\mathbf{q}}_X^h$, $\dot{\mathbf{q}}_Y|_{\mathcal{H}_{\mathbf{q}}} = \dot{\mathbf{q}}_Y^h$.

Proof: For a generic Riemannian submersion, one has (see [11, p. 185])

$$\begin{aligned} \nabla_{\dot{\mathbf{q}}_X^h} \dot{\mathbf{q}}_Y^h &= \nabla_{\rho^{*h}(X_{\rho(\mathbf{q})})} \rho^{*h}(Y_{\rho(\mathbf{q})}) \\ &= \rho^{*h}\left(\overset{\rm dg}{\nabla}_{X_{\rho(\mathbf{q})}} Y_{\rho(\mathbf{q})}\right) \\ &\quad + \frac{1}{2}\left[\rho^{*h}\left(X_{\rho(\mathbf{q})}\right), \rho^{*h}\left(Y_{\rho(\mathbf{q})}\right)\right]^v \end{aligned}$$

but, since $\mathcal{H}$ is integrable, $\left[\rho^{*h}\left(X_{\rho(\mathbf{q})}\right), \rho^{*h}\left(Y_{\rho(\mathbf{q})}\right)\right]$ is horizontal, and the last term disappears. ∎

Corollary III.3: With the same notation as *Proposition III.1*

$$\overset{\rm dg}{\nabla}_{X_{\rho(\mathbf{q})}} Y_{\rho(\mathbf{q})} = J^b(\mathbf{q}) \nabla_{\dot{\mathbf{q}}_X} \dot{\mathbf{q}}_Y. \tag{17}$$

Proof: Because of the integrability of $\mathcal{H}$, pushing (16) to $\mathfrak{se}(3)$, we have

$$\begin{aligned} \overset{\rm dg}{\nabla}_{X_{\rho(\mathbf{q})}} Y_{\rho(\mathbf{q})} &= \rho_*\left(\nabla_{\dot{\mathbf{q}}_X^h} \dot{\mathbf{q}}_Y^h\right) \\ &= \rho_*\left(\nabla_{\dot{\mathbf{q}}_X} \dot{\mathbf{q}}_Y\right) = J^b(\mathbf{q}) \nabla_{\dot{\mathbf{q}}_X} \dot{\mathbf{q}}_Y \end{aligned}$$

which gives us the expression of $\overset{\rm dg}{\nabla}$ in terms of the joint space connection. ∎

A. Workspace Dynamical Equations

The effective Euler–Lagrange equations are the dynamic equations of the robotic chain as "seen" from the end-effector.

Proposition III.2: Under the assumption **A2**, if $\gamma(t) = \rho(\mathbf{q}(t))$, $(d\gamma/dt)|_{\rho(\mathbf{q})} = \gamma X_{\rho(\mathbf{q})}$, and $M_{\text{dg}} = I$, the effective forced Euler–Lagrange equations are

$$\begin{aligned}\dot{\gamma} &= \gamma X_{\rho(\mathbf{q})}\\ \dot{X}_{\rho(\mathbf{q})} &= \text{ad}^*_{X_{\rho(\mathbf{q})}} X_{\rho(\mathbf{q})} + J^b(\mathbf{q})M^{-1}(\mathbf{q})\tau. \end{aligned} \tag{18}$$

Proof: From (17), $\overset{\text{dg}}{\nabla}_{X_{\rho(\mathbf{q})}} X_{\rho(\mathbf{q})} = J^b(\mathbf{q})M^{-1}(\mathbf{q})\tau$. Adding the forcing term to the Euler–Lagrange equations and using left invariance

$$\begin{aligned}\overset{\text{dg}}{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t) &= \gamma\left(\dot{X}_{\rho(\mathbf{q})} - \overset{\text{dg}}{\nabla}_{X_{\rho(\mathbf{q})}} X_{\rho(\mathbf{q})}\right)\\ &= \gamma\left(\dot{X}_{\rho(\mathbf{q})} - M_{\text{dg}}^{-1}\text{ad}^*_{X_{\rho(\mathbf{q})}} M_{\text{dg}} X_{\rho(\mathbf{q})}\right)\\ &= \gamma J^b(\mathbf{q})M^{-1}(\mathbf{q})\tau\end{aligned}$$

and the result follows by writing it as a system of first-order equations. ■

Calling $f = J^b(\mathbf{q})M^{-1}(\mathbf{q})\tau$ the external forces, (18) become the *forced Euler–Poincaré* equations of a mechanical system on $SE(3)$. Notice that just like with $\tilde{\tau}$ in joint space, since $M_{\text{dg}} = I$, f can be intended both as living on $\mathfrak{se}(3)$ or as one forms on $\mathfrak{se}^*(3)$.

B. Pseudoinverse and Horizontal Lift

From (3), the pullback map (14) corresponds to a "pseudoinverse" of $J^b(\mathbf{q})$, i.e., $\rho_*\left(\rho^{*h}\left(X_{\rho(\mathbf{q})}\right)\right) = X_{\rho(\mathbf{q})}$. In robotics, the inverse kinematics of a redundant manipulator is usually based on the Moore–Penrose pseudoinverse

$$\dot{\mathbf{q}}'_X = J^{b^\dagger}(\mathbf{q})X_{\rho(\mathbf{q})}. \tag{19}$$

Call $\tilde{\rho} : (\tilde{\mathbf{Q}}I) \to (\mathbb{R}^6, I)$. Since $T_{\tilde{\mathbf{q}}}\tilde{\mathbf{Q}} \simeq \tilde{\mathbf{Q}}$ and $\tilde{\rho}$ is also a Riemannian submersion, the tangent map $\tilde{\rho}_*|_{\tilde{\mathbf{q}}} : \tilde{\mathbf{Q}} \to \mathbb{R}^6$ is a metric-preserving map between vector spaces of different dimensions, both with Euclidean norm, and therefore, a partial isometry in the language of Appendix A. Hence, $\tilde{\rho}^{*h}|_{\tilde{\mathbf{q}}} = (\tilde{\rho}_*|_{\tilde{\mathbf{q}}})^T$ is both the pullback map giving the horizontal lift of the Riemannian submersion $\tilde{\rho}$ (by definition), and the Moore–Penrose pseudoinverse of the Jacobian map $\tilde{\rho}_*$ by *Proposition A.1*. Under the assumption **A2**, for orientation-preserving maps $\tilde{\rho}$, the singular value decomposition gives $\tilde{\rho}_* = \tilde{P}_6^T\tilde{D}\tilde{P}_n$, where $\tilde{P}_6 \in SO(6)$, $\tilde{P}_n \in SO(n)$, and $\tilde{D}$ is a $6 \times n$ matrix with $\tilde{D}_{ii} = 1$, $i = 1, \ldots 6$, and zero, otherwise. Therefore, $\tilde{\rho}^{*h} = \tilde{P}_n^T\tilde{D}^T\tilde{P}_6$ and

$$\tilde{\rho}^{*h}\tilde{\rho}_* = \tilde{P}_n^T\tilde{D}^T\tilde{P}_6\tilde{P}_6^T\tilde{D}\tilde{P}_n = \tilde{P}_n^T\begin{bmatrix} I_6 & \\ & 0_{n-6}\end{bmatrix}\tilde{P}_n. \tag{20}$$

At each point $\tilde{\mathbf{q}}$, the horizontal and vertical distributions $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{V}}$ of the Riemannian submersion in the $\tilde{\mathbf{Q}}$ basis give orthogonally complementary subspaces of $T_{\tilde{\mathbf{q}}}\tilde{\mathbf{Q}}$, corresponding, respectively, to $\mathcal{N}(\tilde{\rho}^*)^\perp$ and $\mathcal{N}(\tilde{\rho}^*)$ ($\mathcal{N}(\cdot)$ is the null space). Furthermore, since $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{V}}$ are both involutive, the orthogonal matrix $\tilde{P}_n$ has to have a block-diagonal structure

$$\tilde{P}_n = \begin{bmatrix}\tilde{P}_6 & \\ & \tilde{P}_{n-6}\end{bmatrix}, \quad \tilde{P}_6 \in SO(6), \quad \tilde{P}_{n-6} \in SO(n-6). \tag{21}$$

Therefore, (20) becomes simply

$$\tilde{\rho}^{*h}\tilde{\rho}_* = \begin{bmatrix} I_6 & \\ & 0_{n-6}\end{bmatrix}. \tag{22}$$

From $\rho = \varphi \circ \tilde{\rho}$, the chain rule gives

$$J^b(\mathbf{q}) = \rho_*|_{\mathbf{q}} = \tilde{\rho}_*|_{\tilde{\mathbf{q}}=\varphi(\mathbf{q})}\,\varphi_*|_{\mathbf{q}} = \tilde{\rho}_*|_{\tilde{\mathbf{q}}}\,N(\mathbf{q}).$$

The horizontal lift of the robotic chain ρ is then

$$\rho^{*h}\Big|_{\mathbf{q}} = N^{-1}(\tilde{\mathbf{q}})\,\tilde{\rho}^{*h}\Big|_{\tilde{\mathbf{q}}=\varphi(\mathbf{q})}.$$

Theorem III.2: Consider the robotic chain

$$\rho : (\mathbf{Q}, M(\mathbf{q})) \to (SE(3), I).$$

Under the assumption **A2**, the pullback map ρ^{*h} is given by the Moore–Penrose pseudoinverse

$$\rho^{*h}\Big|_{\mathbf{q}} = J^{b^\dagger}(\mathbf{q}) \tag{23}$$

i.e., for the Riemannian submersion ρ, the horizontal lift of any $X \in \mathfrak{se}(3)$ is given by $\dot{\mathbf{q}}_X^h = J^{b^\dagger}(\mathbf{q})X_{\rho(\mathbf{q})}\ \forall\, \mathbf{q} \in \mathbf{Q}$.

Proof: Consider the Riemannian manifold $(\tilde{\mathbf{Q}}, I)$ and replace $\mathfrak{se}(3)$ with $\mathbb{R}^6$ while keeping the same metric tensor I. The passage $(\mathfrak{se}(3), I) \to (\mathbb{R}^6, I)$, although it changes torsion and curvature, is metric preserving, and therefore, causes no harm to the partial isometry property. We use the same symbol $\tilde{\rho}$ for the forward kinematics having Jacobian onto $\mathbb{R}^6$ and onto $\mathfrak{se}(3)$. We claim that $N^{-1}\tilde{\rho}^{*h}$ is the Moore–Penrose inverse for the forward kinematic map $\rho : (\mathbf{Q}, M(\mathbf{q})) \to (SE(3), I)$. To prove it, we need to verify the four properties (I)–(IV) listed in Appendix A, using the fact that analogous properties hold for $\tilde{\rho}^{*h}$ and knowing the structure of $N = D_1 P$

$$\begin{aligned}\text{(I)}:\ \tilde{\rho}_* N N^{-1}\tilde{\rho}^{*h}\tilde{\rho}_* N &= \tilde{\rho}_* N\\ \text{since } \tilde{\rho}_*\tilde{\rho}^{*h}\tilde{\rho}_* &= \tilde{\rho}_*\\ \text{(II)}:\ N^{-1}\tilde{\rho}^{*h}\tilde{\rho}_* N N^{-1}\tilde{\rho}^{*h} &= N^{-1}\tilde{\rho}^{*h}\\ \text{since } \tilde{\rho}^{*h}\tilde{\rho}_*\tilde{\rho}^{*h} &= \tilde{\rho}^{*h}\\ \text{(III)}:\ (\tilde{\rho}_* N N^{-1}\tilde{\rho}^{*h})^T &= \tilde{\rho}_*\tilde{\rho}^{*h}\\ &= \tilde{\rho}_* N N^{-1}\tilde{\rho}^{*h}\\ \text{since } (\tilde{\rho}_*\tilde{\rho}^{*h})^T &= \tilde{\rho}_*\tilde{\rho}^{*h}\\ \text{(IV)}:\ (N^{-1}\tilde{\rho}^{*h}\tilde{\rho}_* N)^T &= N^T(\tilde{\rho}^{*h}\tilde{\rho}_*)^T N^{-T}\\ &= N^T\tilde{\rho}^{*h}\tilde{\rho}_* N^{-T}\\ \text{since } (\tilde{\rho}^{*h}\tilde{\rho}_*)^T &= \tilde{\rho}^{*h}\tilde{\rho}_*.\end{aligned}$$

In order to complete the proof of item (IV) above, we have to show that $N^T\tilde{\rho}^{*h}\tilde{\rho}_* N^{-T} = N^{-1}\tilde{\rho}^{*h}\tilde{\rho}_* N$. From (22) and $N = D_1 P$

$$\begin{aligned}N^T\tilde{\rho}^{*h}\tilde{\rho}_* N^{-T} &= P^T D_1\begin{bmatrix} I_6 & \\ & 0_{n-6}\end{bmatrix}D_1^{-1}P\\ &= P^T D_1^{-1}\begin{bmatrix} I_6 & \\ & 0_{n-6}\end{bmatrix}D_1 P\\ &= N^{-1}\tilde{\rho}^{*h}\tilde{\rho}_* N.\end{aligned}$$

Therefore, from the uniqueness of the Moore–Penrose pseudoinverse

$$J^{b^\dagger}(\mathbf{q}) = \rho^{*h}\Big|_{\mathbf{q}} = N^{-1}(\mathbf{q})\,\tilde{\rho}^{*h}\Big|_{\tilde{\mathbf{q}}} = N^{-1}(\mathbf{q})\left(\tilde{\rho}_h|_{\tilde{\mathbf{q}}}\right).$$

■

Notice that the integrability of $\mathcal{H}$ is crucial to complete the proof of the item (IV) above. In fact, if $\mathcal{H}$ is not integrable, $\tilde{P}_n$ does not admit the block factorization (21), and therefore, $\tilde{\rho}^{*h}\tilde{\rho}_*$ cannot be expressed as (22), since the eigenvalues λ_i of N remain "trapped" between two orthogonal matrices P and $\tilde{P}_n$. $D_1\tilde{P}_n^T\tilde{D}^T\tilde{D}P_n D_1^{-1}$ is then not symmetric anymore, so that $N^{-1}\tilde{\rho}^{*h}$ does not satisfy the property (IV), and it is not possible to reobtain the singular value decomposition of J^b. Obviously, a sufficient condition for local integrability of $\mathcal{H}$ is that $\mathbf{Q}$ is an Euclidean cylinder. It is interesting to remark that the result of *Corollary III.2* is valid for any pseudoinverse $J^{b^\#}$. In [19, Th. 2.1], it is shown that a joint space path is repeatable if and only if all the Lie brackets of vectors formed by the columns of $J^{b^\#}$ belong to span $(J^{b^\#})$. In other words, the six-dimensional surface spanned by the distribution generated by the columns of $J^{b^\#}$ is involutive.

Practical problems of the pseudoinverse scheme in dealing with joint space singularities are well known, and are normally coped with by means of general least squares inverses. This type of solution relaxes the minimum norm property of the Moore–Penrose pseudoinverse, and destroys the partial isometry property.

IV. Application: Horizontal Lift of Workspace Quantities

The geometric notion of horizontal lift allows having an intrinsic "preferred" way of mapping back to joint space interesting workspace quantities. In particular, any function $\theta : SE(3) \rightarrow \mathbb{R}$ induces a unique function in joint space, call it $\vartheta : \mathbf{Q} \rightarrow \mathbb{R}$ with the same dynamics as θ. It is obtained as follows. In $\gamma \in SE(3)$, there exists a unique Lie algebra-valued covector $\eta(t) \in \mathfrak{se}^*(3) \simeq \mathbb{R}^6$ corresponding to $\mathrm{d}\theta|_{\gamma}(t)$ (since $M_{\mathrm{dg}} = I$, it also coincides with $\mathrm{grad}\,\theta(\gamma(t))$), defined as $\eta(t) = \gamma^{-1}(t)\ \mathrm{d}\theta|_{\gamma(t)}$. Then, from *Corollary III.1*, if we fix a point $\mathbf{q}_0$ in the fiber at $\gamma(0)$, $\mathbf{q}_0 \in \rho^{-1}(\gamma(0))$, we obtain a unique real-valued function $\vartheta(\mathbf{q}(t))$ "attached" at $\mathbf{q}_0$ for $t = 0$. In fact, similar to (14), there exists a unique covector $\mathrm{d}\vartheta(\mathbf{q}) = \rho^{*h}\big|_{\mathbf{q}}\, \eta = J^{b^\dagger}(\mathbf{q})\eta$, and the gradient of $\vartheta(\mathbf{q})$ is uniquely given by

$$\mathrm{grad}\vartheta(\mathbf{q}) = M^{-1}(\mathbf{q}) J^{b^\dagger}(\mathbf{q}) \gamma^{-1} \mathrm{grad}\theta(\mathbf{q}) \tag{24}$$

from which $\vartheta(\mathbf{q})$ is obtained by numerical quadrature, starting from the initial condition $\vartheta(\mathbf{q}_0)$.

The function θ could represent the result of a distance measurement in the workspace, typical examples being the case of visual servoing, or a force/torque measurement on the end-effector.

As the Riemannian submersion respects the Lie group structure of the workspace, the geometric formulation carried out in this paper could be used to resolve the redundancy, while enabling using the geometric methods for trajectory generation [1], [21] and tracking [2], [9], [15] to design more effective motion-control algorithms for robotic manipulators directly in the workspace. In this case, the horizontal lift assures that the error dynamics of the pulled back joint space feedback controller is identical to that of the original workspace feedback.

V. Conclusion

For "flat" redundant robotic chains composed of simple 1-DOF joints or links, a geometric interpretation of the forward kinematic map in terms of Riemannian submersions is proposed. Several properties of redundant robots then admit clear geometric characterizations, the most remarkable being that the Moore–Penrose pseudoinverse normally used in robotics coincides with the horizontal lift of the Riemannian submersion. Then, many known algorithms of common use in robotics acquire also an intrinsic geometric meaning.

Appendix

A. Generalized Inverses and Partial Isometries

We need a few facts from the theory of generalized inverse of rectangular matrices, see, for example, [7]. For a given $m \times n$ real matrix A, a pseudoinverse is a matrix B characterized by one or more of the four defining properties

$$\begin{aligned} ABA &= A \quad \text{(I)} \\ BAB &= B \quad \text{(II)} \\ (AB)^T &= AB \quad \text{(III)} \\ (BA)^T &= BA \quad \text{(IV)}. \end{aligned}$$

Each of the properties (I)-(IV) defines a particular type of pseudoinverse of A. The "weakest" pseudoinverse will satisfy only the first property (each such B is indicated as $B^{\{I\}}$); on the other end of the scale, the "strongest" inverse is the Moore–Penrose pseudoinverse $B^\dagger = B^{\{\mathrm{I,II,III,IV}\}}$, i.e., the unique matrix satisfying all four properties above.

A linear transformation between vector spaces equipped with Euclidean norms $A : V_x \rightarrow V_b$ is called a *partial isometry* if it is norm preserving on the orthogonal complement of its null space $\mathcal{N}(A)$, i.e., if $\|Ax\| = \|x\|\ \forall\, x \in \mathcal{N}(A)^\perp$ or, equivalently, if it is distance preserving $\|Ax_1 - Ax_2\| = \|x_1 - x_2\|\ \forall\, x_1, x_2 \in \mathcal{N}(A)^\perp$. So a partial isometry is basically an isometry maintained through embeddings or submersions. For partial isometries we have the following characterization.

Proposition A.1: ([7, Ch.6, Th. 4]) The following statements are equivalent:

i) $A \in \mathbb{R}^{m\times n}$ is a partial isometry;
ii) A^T is a partial isometry;
iii) $AA^TA = A$ and $A^TAA^T = A^T$;
iv) $A^\dagger = A^T$.

A consequence is that all the nonzero eigenvalues of the symmetric matrix A^TA are unitary (equal to one for orientation-preserving maps).

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An Analytical Expression for the Generalized Forces in Multibody Lagrange Equations

Patrick Kool

***Abstract*—This paper describes how the partial derivative of the kinetic energy, with respect to the generalized coordinates in the Lagrange equations, can be obtained in analytic form for structures consisting of rigid links connected by lower pair joints. We will prove that the expression for the derivative involves the time derivative of the line coordinates of the geometric lines, coinciding with the joint axes. As a consequence, the generalized force in the Lagrange equations can be written as a function of the inertia matrices and the line coordinates of the joint axes. The time derivative of the line coordinates can be expressed by using the adjoint matrix of the line vector.**

***Index Terms*—Lagrange equation, line coordinates, multibody dynamics.**

I. Introduction

The Lagrange equation based on generalized coordinates has been used successfully to derive the dynamic equations of multibody structures, and for serial robots, in particular [1]. This approach is useful when one is interested in the generalized forces only. To obtain the reaction forces and torques, the momentum equations have to be used. Concise formulations of the dynamics [2] based on these equations have been developed with dual vectors, using the concepts of twists and wrenches. The utility of screws (complex vectors) written as dual vectors to derive rigid-body dynamics has been stressed early in [3]. In [4], a dual Lagrange equation is formulated by developing derivative rules with respect to dual variables. In [5], both a recursive Newton–Euler and a closed-form Lagrangian formulation based on a Lie group formulation is presented. Using the virtual work and D'Alembert's principles, the inverse dynamics can be obtained in an analytic form [6]. In this paper, line coordinates and spatial line transforms [7] will be used to derive the Lagrange equations without using partial derivatives of the kinetic energy. From the Lagrange equations, we will derive an analytical expression for the partial derivative of the kinetic energy, with respect to the generalized coordinates. As a result, the generalized force will be expressed in analytic form as a function of the inertia matrices and the line vectors representing the twist in the joint axes.

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The author is with the Department of Mechanics, Vrije Universiteit Brussel, B-1050 Brussels, Belgium (e-mail: pkool@vub.ac.be).

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II. Multibody Dynamics

Considering a chain of rigid bodies connected by revolute, prismatic, or helicoidal joints, the Lagrange equation for link i with the generalized momentum p_i reads

$$\tau_i = \frac{dp_i}{dt} - \frac{\partial E}{\partial q_i}. \tag{1}$$

The joint variables are chosen as generalized coordinates q_i. The kinetic energy in a chain with n links is given by

$$E = \sum_{j=1}^{n} .5 T_j^T N_j T_j. \tag{2}$$

The inertia matrix [8] of link j is given by

$$N_j = \begin{bmatrix} I & m\bar{r}_g^\star \\ -m\bar{r}_g^\star & mE_3 \end{bmatrix}_j. \tag{3}$$

The adjoint $\bar{r}_g^\star$ of the position vector of the center of gravity is given in the absolute reference frame, and the inertia submatrix I is given with respect to the origin of this frame. The matrix E_3 is a unit matrix. As the inertia matrix is symmetric, the generalized momentum p_i, given by the partial derivative of the kinetic energy, with respect to the generalized velocity $\dot{q}_i$, is

$$p_i = \frac{\partial E}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \sum_{j=1}^{n} .5 T_j^T N_j T_j = \sum_{j=1}^{n} T_j^T N_j \frac{\partial T_j}{\partial \dot{q}_i}. \tag{4}$$

The twist vector T_k of each link k can be written with the Jacobian matrix, but it is more useful to stress the fact that the columns of the Jacobian matrix are the line vectors L of the axes of the joints [9], hence

$$T_k = \sum_{j=1}^{k} L_{j-1} \dot{q}_j. \tag{5}$$

The first axis has index zero and is stationary, while an axis with index j represents the axis fixed to link j. For a helicoidal joint with pitch h, the line vector is given by

$$L_{i-1,h} = \begin{bmatrix} \bar{e}_{i-1} \\ \bar{m}_{i-1} + h\bar{e}_{i-1} \end{bmatrix}. \tag{6}$$

The unit vector $\bar{e}$ along the joint axis is given in the absolute reference frame, and the moment $\bar{m}$ of the line is calculated from the origin of this frame, and is projected in the same frame. To obtain the line coordinates for a revolute joint, the pitch h must be set to zero. For a prismatic joint, the line coordinates just contain the direction cosines of the unit vector along the joint axis, and this corresponds to an infinite pitch

$$L_{i-1,P} = \begin{bmatrix} \bar{0} \\ \bar{e}_{i-1} \end{bmatrix}. \tag{7}$$

From (5) and the symmetry of the inertia matrix then follows after transposition for $j \geq i$, denoting the sum of the linear and angular momentum from link i till the end of the chain as M_i

$$p_i = \left(\sum_{j=i}^{n} T_j^T N_j \right) L_{i-1} = L_{i-1}^T \left(\sum_{j=i}^{n} N_j T_j \right) = L_{i-1}^T M_i. \tag{8}$$

Observe that with (5), the generalized momentum p_i is a function of the inertia matrices and the line vectors. The time derivative of the generalized momentum is

$$\frac{dp_i}{dt} = L_{i-1}^T \frac{dM_i}{dt} + \dot{L}_{i-1}^T M_i. \tag{9}$$

The equation of motion (1) becomes

$$\tau_i = L_{i-1}^T \frac{dM_i}{dt} + \dot{L}_{i-1}^T M_i - \frac{\partial E}{\partial q_i}. \tag{10}$$

The power delivered by the generalized force τ_i with the generalized velocity $\dot{q}_i$ equals the reciprocal product [9], [10] of the local twist