Consensus problems on networks with antagonistic interactions

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***Abstract*—In a consensus protocol an agreement among agents is achieved thanks to the collaborative efforts of all agents, expresses by a communication graph with nonnegative weights. The question we ask in this paper is the following: is it possible to achieve a form of agreement also in presence of antagonistic interactions, modeled as negative weights on the communication graph? The answer to this question is affirmative: on signed networks all agents can converge to a consensus value which is the same for all agents except for the sign. Necessary and sufficient conditions are obtained to describe the cases when this is possible. These conditions have strong analogies with the theory of monotone systems. Linear and nonlinear Laplacian feedback designs are proposed.**

Index terms – Consensus protocols; Signed graphs; Structural balance; Monotone Systems.

I. Introduction

The problem of reaching a consensus among a group of agents using only local actions has a long history [15], [13] and in recent years it has received a remarkable attention from different perspectives, thanks to the large number of potential applications, ranging from Engineering and Computer Science (distributed computation [6], sensory networks, formation control of mobile robots [32]) to Biology, Ecology and Social Sciences (self-driven motion of biological particles [46], collective behavior of flocks and herds [41], dynamics of opinion forming [21]). See [40], [36] for a more organic overview of the field.

A common *trait* of basically all the current research on the consensus problem is the focus on *cooperative systems*. Consensus in these systems is achieved through collaboration: the network of interactions representing the communications between agents is characterized by edge weights that are nonnegative. In several real world scenarios, however, it is more plausible to assume that some agents collaborate, while other *compete*. Networks with antagonistic interactions are common for example in social network theory [47], [16]. They are represented as signed graphs, i.e., graphs in which the edges can assume also negative weights. A positive/negative weight can be associated to a friend/foe (allied/adversary) relationship between the two agents linked by the edge, or, depending on the context, to a trust/distrust, like/dislike, etc. interaction, see [47], [16], [18].

Our aim in this paper is to introduce a suitable notion of consensus in presence of antagonistic links and to investigate how and to what extent agents on signed graphs can achieve consensus through distributed protocols. In particular, we will see that under suitable conditions the agents can achieve a form of "agreed upon dissensus" (hereafter called *bipartite consensus*), in which all agents converge to a value which is the same for all in modulus but not in sign. This polarization of the community into two factions characterized by opposite "opinions" is common in many antagonistic systems describing bimodal coalitions, like two-party political systems, duopolistic markets, rival business cartels, competing international alliances (think of the Iron curtain era), etc. See [47], [16] for more details on applications in social networks theory. Potential engineering applications are also easily conceivable (even beyond warfare scenarios). Some of these applications (such as trust networks) are mentioned in [17], [29], where attempts to deal with signed graphs are made (see also [4]). We will show that if we use distributed Laplacian-like schemes as in the current literature on consensus problems, then bipartite consensus can be achieved when and only when the signed graph of the network is *structurally balanced*. In social network theory, structural balance is a well-known property [19], [12], and corresponds to the possibility of exactly bipartitioning the signed graph into two adversary subcommunities such that all edges within each subcommunity have positive weights while all edges joining agents of different communities have negative weights. Graphs of nonnegative weights are a special case of structural balance, in which one of the two subcommunities is empty.

We will show that Laplacian schemes are convergent also on signed graphs that are not structurally balanced (provided the Laplacian is defined properly). However, in this case the consensus value is always trivial (the origin), regardless of the initial condition and of the antagonistic content of the network. In fact, the Laplacian one obtains in the structurally unbalanced case is globally asymptotically stable (rather than critically stable), meaning that the (bipartite for us) agreement subspace is empty. It is worth observing that the asymptotic stability of the family of Laplacians corresponding to structurally unbalanced graphs cannot be explained by standard stability arguments, such as the analysis of the Geršgorin disks or diagonal dominance [30], [22].

An equivalent characterization of structurally balance signed graphs is that all cycles (or semicycles for directed graphs) of the graph are positive, i.e., have an even number of negative edges. Quite remarkably, this condition is formally analogous to the so-called Kamke condition for Jacobians of monotone systems [43]. The analogy can be made rigorous by observing that all structurally balanced networks are equivalent, under a suitable change of orthant order, to nonnegative networks. Adopting the terminology used in Statistical Physics for this type of equivalence transformations (well-known in the Ising spin glass literature [7], see [28], [2], [18] for more details), we shall call the changes of orthant order *gauge transformations*. Gauge equivalence (or switching equivalence in the theory of signed graphs [48], or signature similarity as it is called in the field of signed pattern matrices [10]) is a finite-cardinality subclass of the similarity equivalence of matrices, which leaves the modulus of the entries of a matrix unchanged and only modifies its sign pattern. Given a structurally balanced network with its set of edge weights, there exists a family of structurally balanced signed networks characterized by the same weights (but with different signs). All these "realizations" of the signed networks are related by gauge transformations and all are isospectral, meaning that the corresponding Laplacians enjoy the same convergence properties, although the bipartition characterizing the consensus vector differs from realization to realization. In particular, in each such family of gauge equivalent structurally balanced networks there is always one particular network with all nonnegative weights. It is therefore possible to adapt both linear [38] and nonlinear [39], [27], [26], [34], [45] Laplacian schemes used for "standard" consensus to the case of structurally balanced networks.

That Laplacian feedback schemes for consensus correspond to contractions in a proper metric space, and that these contractions are a special case of monotone systems was observed already by L. Moreau [37]. Expanding on this observation, here we show that any monotone system can be turned into a (nonlinear) Laplacian scheme achieving bipartite consensus.

The rest of the paper is organized as follows: basic definitions and properties of signed graphs are recalled in Section II; linear consensus protocols for undirected and directed signed graphs are discussed in Section III, while several nonlinear consensus protocols are presented in Section IV.

II. Signed graphs

A (weighted) signed graph $\mathcal{G}$ is a triple $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$ where $\mathcal{V} = \{v_1, \ldots, v_n\}$ is a set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges, and $A \in \mathbb{R}^{n\times n}$ is the matrix of the signed weights of $\mathcal{G}$: $a_{ij} \neq 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}$. The adjacency matrix A alone completely specifies a signed graph. For the signed graph corresponding to A we shall use the notation $\mathcal{G}(A)$. We will not consider graphs with self-loops: $a_{ii} = 0 \;\forall\, i = 1, \ldots, n$. When the graph is undirected then the order of the nodes in $\mathcal{E}$ is irrelevant and the matrix A is symmetric. For a directed graph (digraph) we shall use the convention that on the edge $(v_j, v_i) \in \mathcal{E}$, v_j represents the tail and v_i the head of the arrow. In a digraph a pair of edges sharing the same nodes (v_i, v_j), $(v_j, v_i) \in \mathcal{E}$ is called a digon. In the digraphs of this paper we will always assume that $a_{ij}a_{ji} \geqslant 0$, meaning that the edge pairs of all digons cannot have opposite signs. Under this assumption (hereafter called *digon sign-symmetry*), a digraph $\mathcal{G}$ "admits" an undirected graph $\mathcal{G}(A_u)$ defined by $A_u = (A + A^T)/2$.

A (directed) path $\mathcal{P}$ of $\mathcal{G}(A)$ is a concatenation of (directed) edges of $\mathcal{E}$:

$$\mathcal{P} = \{(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_{p-1}}, v_{i_p})\} \subset \mathcal{E}$$

in which all nodes $v_{i_1}, \ldots, v_{i_p}$ are distinct. The length of $\mathcal{P}$ is $p - 1$. A (directed) *cycle* $\mathcal{C}$ of $\mathcal{G}(A)$ is a (directed) path beginning and ending with the same node $v_{i_p} = v_{i_1}$. For digraphs, a semicycle is a cycle of $\mathcal{G}(A_u)$. A cycle (semicycle) is positive if it contains an even number of negative edge weights: $a_{i_1,i_2} \ldots a_{i_p,i_1} > 0$. It is negative if $a_{i_1,i_2} \ldots a_{i_p,i_1} < 0$. Irreducibility of A corresponds to $\mathcal{G}(A)$ which is strongly connected, i.e., $\forall\, v_i, v_j \in \mathcal{V} \;\exists\; \mathcal{P} \subset \mathcal{E}$ starting at v_i and ending at v_j (strong connectivity collapses into connectivity when A is symmetric).

The following is mentioned in e.g. [44] but not proved. We therefore provide a self-contained proof in the Appendix.

Proposition 1 *Consider a digraph $\mathcal{G}(A)$ which is strongly connected and digon sign-symmetric. $\mathcal{G}(A)$ has no negative semicycle if and only if $\mathcal{G}(A)$ has no negative directed cycle.*

Given the signed digraph $\mathcal{G}(A)$, denote C_r the row connectivity matrix of A, i.e., the diagonal matrix having diagonal elements $c_{r,ii} = \sum_{j\in \mathrm{adj}(i)} |a_{ij}|$, where $\mathrm{adj}(i)$ are the nodes adjacent to v_i in $\mathcal{E}$ (with in-degree direction as in [36]: v_j is the tail of the arrow whose head is v_i). The column connectivity matrix C_c is defined analogously. When $A = A^T$ then $C_r = C_c = C$. More generally, a signed digraph is said *weight balanced* if $C_r = C_c$. In the consensus literature [36], [40], [38], this property is normally referred to as "balanced" *tout-court* (see [14], though). For matrices of nonnegative weights, it can also be expressed as

$$A\mathbf{1} = A^T\mathbf{1}, \tag{1}$$

where $\mathbf{1} = \begin{bmatrix}1 \ldots 1\end{bmatrix}^T \in \mathbb{R}^n$.

III. Linear consensus protocols for signed graphs

Consider the system of integrators

$$\dot{x} = u, \qquad x,\; u \in \mathbb{R}^n. \tag{2}$$

In the consensus problem, the task is to devise distributed feedback laws $u_i = u_i(x_i, x_j, j \in \mathrm{adj}(i))$, $i = 1, \ldots, n$, i.e., feedback laws based on the states of the node itself and of its first neighbors on the connectivity graph $\mathcal{G}(A)$ of the network. Unlike in standard consensus problems, we do not assume that the weights of A are nonnegative.

A. Undirected graphs

Consider a given signed (symmetric) adjacency matrix A. The definition of a Laplacian L in the case of signed A is $L = C - A$ where in the connectivity matrix C the weights are in absolute value, see [24], [31][1]. The elements of L are

[1]This is not the only definition of Laplacian of a signed graph available in the literature. In [8], for example, the Laplacian is defined without the absolute values in the diagonal terms. In this formulation 0 is always an eigenvalue, but negative eigenvalues may appear, rendering the Laplacian useless for convergence purposes.

therefore:

$$\ell_{ik} = \begin{cases} \sum_{j \in \text{adj}(i)} |a_{ij}| & k = i \\ -a_{ik} & k \neq i. \end{cases}$$

The corresponding Laplacian potential is

$$\begin{aligned} \Phi(x) = x^T L x &= \sum_{(v_j, v_i) \in \mathcal{E}} \left(|a_{ij}| x_i^2 + |a_{ij}| x_j^2 - 2 a_{ij} x_i x_j \right) \\ &= \sum_{(v_j, v_i) \in \mathcal{E}} |a_{ij}| \left(x_i - \text{sgn}(a_{ij}) x_j \right)^2 , \end{aligned} \tag{3}$$

where $\text{sgn}(\cdot)$ is the sign function. The effect of a negative weight a_{ij} is to replace the usual $(x_i - x_j)^2$ term in (3) with $(x_i + x_j)^2$, which does not alter the sum of squares structure of $\Phi(x)$.

Just like for the nonnegative weights case, one can use L for the feedback laws in (2) and study the gradient system

$$\dot{x} = -Lx, \tag{4}$$

which in components reads:

$$\dot{x}_i = - \sum_{j \in \text{adj}(i)} |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j).$$

Let $\lambda_1(L) \leqslant \ldots \leqslant \lambda_n(L)$ be the eigenvalues of L. From (3) it is evident that $\Phi(x) \geqslant 0$ and therefore that $\lambda_1(L) \geqslant 0$. Unlike for the case of nonnegative A, here $-L$ is however no longer a Metzler matrix[2] in general and its row/column sum need not be zero. The major difference with the standard theory of nonnegative adjacency matrices is that L can be positive definite.

Example 1 Consider the signed graph of Fig. 1(a) of adjacency matrix

$$A_1 = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -4 \\ -2 & -4 & 0 \end{bmatrix},$$

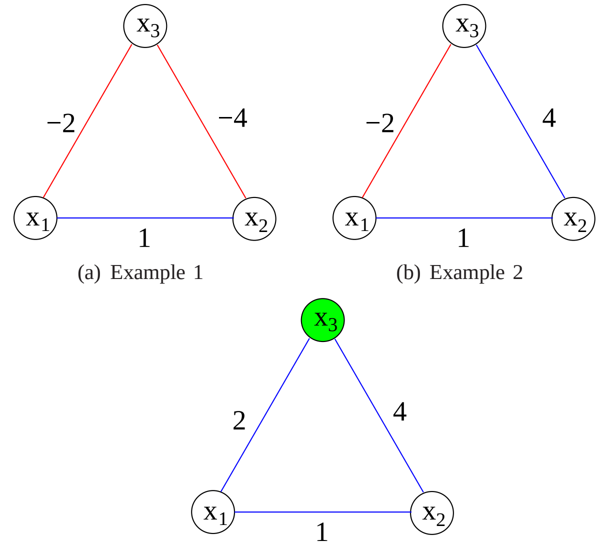
the corresponding Laplacian $L_1 = \text{diag}(3, 5, 6) - A_1$ has eigenvalues $\text{sp}(L_1) = \{0,\ 4.35,\ 9.65\}$, i.e. L_1 positive semidefinite.

Example 2 The signed graph of Fig. 1(b) instead has adjacency matrix

$$A_2 = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 4 \\ -2 & 4 & 0 \end{bmatrix}.$$

The Laplacian $L_2 = \text{diag}(3, 5, 6) - A_2$ has eigenvalues $\text{sp}(L_2) = \{1.2,\ 2.61,\ 10.18\}$, meaning that L_2 positive definite.

[2]Metzler matrices, also called negated Z-matrices, are matrices with nonnegative off-diagonal entries, see [5].



(a) Example 1

(b) Example 2

(c) Example 3

Fig. 1. Signed undirected connectivity graphs mentioned in Section III-A. Examples 1 and 3 are structurally balanced and differ only by the gauge transformation $D = \text{diag}(1,\ 1,\ -1)$. Example 2 is structurally unbalanced.

1) Effect of a gauge transformation: A partial orthant order in $\mathbb{R}^n$ is a vector $\sigma = \begin{bmatrix} \sigma_1 & \ldots \sigma_n \end{bmatrix}$, $\sigma_i \in \{\pm 1\}$. A *gauge transformation* is a change of orthant order in $\mathbb{R}^n$ performed by a matrix $D = \text{diag}(\sigma)$. Denote $\mathcal{D} = \{D = \text{diag}(\sigma),\ \sigma = \begin{bmatrix} \sigma_1 \ldots \sigma_n \end{bmatrix},\ \sigma_i \in \{\pm 1\}\}$ the set of all gauge transformations in $\mathbb{R}^n$. Given the system (4), consider the change of coordinates corresponding to the gauge transformation D:

$$z = Dx, \qquad D \in \mathcal{D}. \tag{5}$$

Since $D^{-1} = D$, $x = Dz$, and from (4)

$$\dot{z} = -L_D z, \tag{6}$$

where $L_D = DLD = C - DAD$ is the new Laplacian of the gauge transformed system. In components,

$$\ell_{D,ik} = \begin{cases} \sum_{j \in \text{adj}(i)} |a_{ij}| & k = i \\ -\sigma_i \sigma_k a_{ik} & k \neq i. \end{cases}$$

Proposition 2 *L and L_D are isospectral:* $\text{sp}(L) = \text{sp}(L_D)$. *The class of gauge equivalent Laplacians $\mathcal{L}(L) = \{DLD,\ D \in \mathcal{D}\}$ contains at most 2^{n-1} distinct matrices.*

Proof: $D \in \mathcal{D}$ is such that $|\det D| = 1$, $D^{-1} = D = D^T$. Hence the transformation $L \to DLD$ is a similarity transformation and as such it preserves the spectrum. The set $\mathcal{D}$ contains 2^n diagonal matrices D and each corresponding gauge transformation changes the signs of the rows/columns corresponding to the -1 entries of D. When L connected, all L_D in $\mathcal{L}(L)$ are distinct, up to a global symmetry: $DLD = (-D)L(-D)$. ∎

It follows from Proposition 2 that also $\text{sp}(A) = \text{sp}(DAD)$.

Example 3 Applying the gauge transformation $D = \mathrm{diag}(1,\,1,\,-1)$ to A_1 of Example 1 one gets

$$A_3 = DA_1D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix},$$

i.e., a nonnegative adjacency matrix isospectral with A_1, see Fig. 1(c). The corresponding $L_3 = \mathrm{diag}(3,5,6) - A_3$ can therefore be used in (6) to solve a standard average consensus problem. In this case $\ker(L) = \mathrm{span}(\mathbf{1})$ is the agreement subspace and, following [38], the solution of the average consensus problem is

$$z^* = \lim_{t\to\infty} z(t) = \frac{1}{n}(\mathbf{1}^T z(0))\mathbf{1}. \tag{7}$$

2) Structural balance and bipartite consensus: Since Example 3 is a standard consensus problem and since $\mathrm{sp}(L_1) = \mathrm{sp}(L_3)$, it is intuitively clear that also for Example 1 a consensus problem can be formulated and that its solution x^* must be related to z^* of Example 3. In particular, from (5), $x_i^* = z_i^*$ $i = 1, 2$, $x_3^* = -z_3^*$, i.e., $|x| = |z|$, meaning that the components of x converge to values which agree in modulus but differ in sign. This asymptotic behavior is a form of "agreed dissensus", which we shall denote *bipartite consensus*. More formally, we have:

Definition 1 *The system* (4) *admits a bipartite consensus solution if* $\lim_{t\to\infty} |x_i(t)| = \alpha > 0\ \forall\, i = 1, \ldots, n$.

It is not too difficult to verify that no gauge transformation $D \in \mathcal{D}$ exist able to render DA_2D nonnegative. In order to understand the difference between Example 1 (and 3) and Example 2, it is useful to introduce the notion of structurally balance signed network and its equivalence characterizations.

Definition 2 *A signed graph* $\mathcal{G}(A)$ *is said* structurally balanced *if it admits a bipartition of the nodes* $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = 0$ *such that* $a_{ij} \geqslant 0\ \forall\, v_i,\, v_j \in \mathcal{V}_q$ $(q \in \{1,\,2\})$, $a_{ij} \leqslant 0\ \forall\, v_i \in \mathcal{V}_q$, $v_j \in \mathcal{V}_r$, $q \neq r$ $(q,\, r \in \{1,\,2\})$. *It is said* structurally unbalanced *otherwise.*

Lemma 1 *A connected signed graph* $\mathcal{G}(A)$ *is structurally balanced if and only if any of the following equivalent conditions holds:*

1) *all cycles of* $\mathcal{G}(A)$ *are positive;*
2) $\exists\ D \in \mathcal{D}$ *such that* DAD *has all nonnegative entries;*
3) 0 *is an eigenvalue of* L.

Proof:

1) This is a classical result from [12][3];
2) From Definition 2, $\mathcal{V}$ can be partitioned such that all and only the negative edges have a node in $\mathcal{V}_1$ and the other in $\mathcal{V}_2$. It is enough to choose $D = \mathrm{diag}(\sigma)$ with σ such that $\sigma_i = +1$ when $v_i \in \mathcal{V}_1$ and $\sigma_i = -1$ when $v_i \in \mathcal{V}_2$ to attain the sought gauge transformed adjacency matrix DAD with all nonnegative entries.
3) If A is structurally balanced then $\exists\ D \in \mathcal{D}$ such that DAD is nonnegative. Therefore the corresponding Laplacian $C - DAD$ has 0 as eigenvalue, and by Proposition 2 so does the Laplacian $L = C - A$. To prove the converse assume $\lambda_1(L) = 0$. Since A is symmetric, $\exists\, w \in \mathbb{R}^n$, $w \neq 0$, such that $Lw = w^T L = 0$, i.e., w is a left and right eigenvector of L. By contradiction, assume A has at least a negative cycle $\mathcal{C} = \{(v_{i_1}, v_{i_2}), \ldots, (v_{i_p}, v_{i_1})\} \subseteq \mathcal{E}$ such that $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_p i_1} < 0$. From (3), the Laplacian potential $\Phi(x)$ can be split accordingly:

$$\begin{aligned}\Phi(x) = &\sum_{(v_i,v_j)\in\mathcal{C}} |a_{ij}|(x_i - \mathrm{sgn}(a_{ij})x_j)^2 \\ &+ \sum_{(v_i,v_j)\in\mathcal{E}\setminus\mathcal{C}} |a_{ij}|(x_i - \mathrm{sgn}(a_{ij})x_j)^2.\end{aligned} \tag{8}$$

Let us focus on the first summation. Without loss of generality, assume only one of the a_{ij} edges of $\mathcal{C}$ has negative weight (since $\mathcal{C}$ has an odd number of negative edges and each node intersects $\mathcal{C}$ in at most two edges, it is always possible to find a $D \in \mathcal{D}$ such that only one negative edge is left in $\mathcal{C}$; all our considerations are invariant to gauge transformations). Assume for example that $a_{i_1 i_2} > 0, \ldots, a_{i_{p-1} i_p} > 0$ and $a_{i_p i_1} < 0$. From $w^T L w = 0$, owing to the sum of square form of $\Phi(x)$, each term in (8) must be 0 in correspondence of w. In particular, expanding the first summation in (8)

$$\begin{aligned}a_{i_1 i_2}(w_{i_1} - w_{i_2})^2 + \ldots + a_{i_{p-1} i_p}(w_{i_{p-1}} - w_{i_p})^2 \\ + |a_{i_p i_1}|(w_{i_p} + w_{i_1})^2 = 0.\end{aligned} \tag{9}$$

From the first $p-1$ terms of (9) we deduce $w_{i_1} = w_{i_2} = \ldots = w_{i_p}$. But this implies that the last term in (9) cannot be zero unless $w_{i_1} = \ldots = w_{i_p} = 0$. Consider now $\mathcal{V}_{\mathcal{C}} = \{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$ and its complement in $\mathcal{V}$: $\hat{\mathcal{V}}_{\mathcal{C}} = \mathcal{V} \setminus \mathcal{V}_{\mathcal{C}}$. Owing to the connectivity of $\mathcal{G}(A)$, it is always possible to find a collection of paths in $\mathcal{G}(A)$ linking all nodes of $\hat{\mathcal{V}}_{\mathcal{C}}$ to those of $\mathcal{V}_{\mathcal{C}}$. Let $\mathcal{P} = \{(v_{j_1}, v_{j_2}), \ldots, (v_{j_q}, v_{i_k})\} \subset \mathcal{E}$ with $v_{j_1}, v_{j_2}, \ldots, v_{j_q} \in \hat{\mathcal{V}}_{\mathcal{C}}$ and $v_{i_k} \in \mathcal{V}_{\mathcal{C}}$. When $x = w$ and $\Phi(w) = 0$, from (8) and $w_{i_k} = 0$ it follows that $w_{j_1} = \ldots = w_{j_q} = 0$. Iterating the argument until all nodes of $\hat{\mathcal{V}}_{\mathcal{C}}$ are covered, we obtain $w = 0$, and hence we have a contradiction.

■

Remark 1 The key argument for the absence of the 0 eigenvalue in structurally unbalanced Laplacians is the impossibility of satisfying all the constraints imposed by $\Phi(x) = 0$ by choosing a combination of signs of the variables x_i. When such a combination of sign exists then we have structural balance.

This argument can be readily applied to spanning trees.

Corollary 1 *A spanning tree is always structurally balanced.*

Proof: When $\mathcal{G}(A)$ is a spanning tree no cycle is present, and, for each signature of the $n-1$ edges a_{ij}, $\mathcal{G}(A)$ has n

[3] This condition is often taken as definition of structural balance [12], [16]. For our purposes, the bipartition of Definition 2 is more evocative as definition.

variables available in order to fulfill the condition $\Phi(x) = 0$ mentioned in Remark 1. ∎

From conditions 2 and 3 of Lemma 1, it follows that on a structurally balanced graph L is positive semidefinite and $\ker(L) = \mathrm{span}(D\mathbf{1})$. Lemma 1 induces also a characterization of structurally unbalanced graphs.

Corollary 2 *A connected signed graph $\mathcal{G}(A)$ is structurally unbalanced if and only if any of the following equivalent conditions holds:*

1) *one or more cycles of $\mathcal{G}(A)$ are negative;*
2) *$\nexists\, D \in \mathcal{D}$ such that DAD has all nonnegative entries;*
3) *$\lambda_1(L) > 0$ i.e., $\Phi(x) > 0$.*

Proof: Since structural balance and unbalance are mutually exclusive properties, the 3 conditions (and their equivalence) follow straightforwardly from Lemma 1. ∎

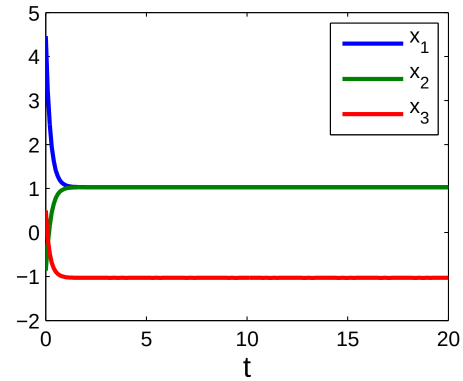
In particular, condition 3) implies that for the structurally unbalanced case $\ker(L) = \{0\}$. This, together with Lemma 1, gives the conditions required to solve the bipartite consensus problem.

Theorem 1 *Consider a connected signed graph $\mathcal{G}(A)$. The system* (4) *admits a bipartite consensus solution if and only if $\mathcal{G}(A)$ is structurally balanced. If $D \in \mathcal{D}$ is the gauge transformation that renders DAD nonnegative, then the bipartite solution of* (4) *is $\lim_{t\to\infty} x(t) = \frac{1}{n}\left(\mathbf{1}^T D x(0)\right) D\mathbf{1}$. If instead $\mathcal{G}(A)$ is structurally unbalanced then $\lim_{t\to\infty} x(t) = 0$ $\forall\, x(0) \in \mathbb{R}^n$.*

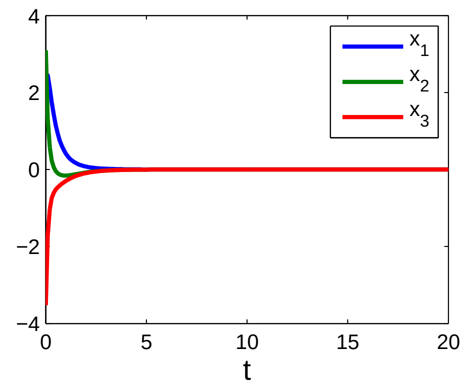
Proof: The first part follows straightforwardly from condition 3 of Lemma 1. The second from the observation that for the gauge transformed system $z = Dx$ the problem is a usual average consensus problem on an undirected, connected graph, whose solution is (7). That such a D exists is guaranteed by condition 2 of Lemma 1. In the structurally unbalanced case, the Laplacian potential $\Phi(x)$ is positive definite, which implies the last sentence. ∎

A comparison of the steady state values reached in the Examples 1-3 is shown in Fig. 2. The gauge transformation $D = \mathrm{diag}(1,\ 1,\ -1)$ allows to pass from Example 1 (bipartite consensus) to Example 3 (standard consensus).

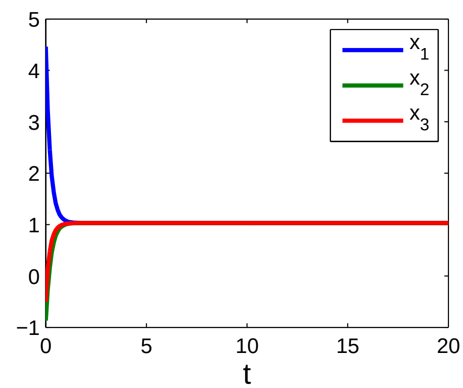
3) A complete classification in gauge equivalent classes: Assume the symmetric adjacency matrix A is given. Assume A has $2m$, $2(n-1) \leqslant 2m \leqslant n^2 - n$, nonzero entries (i.e., $\mathcal{G}(A)$ has m undirected edges) and that the entries of A are given only in modulus. As we vary the signs of the m edges, we have 2^m possible signed graphs, and hence 2^m distinct Laplacians L. From Proposition 2, in the connected case each gauge equivalence class contains 2^{n-1} distinct elements, hence the 2^m signed graphs split into 2^{m-n+1} equivalence classes, each characterized by a different spectrum. From Lemma 1 and Corollary 2, in only one of these classes L is positive semidefinite, while in all the other $2^{m-n+1} - 1$ classes L is positive definite.



(a) Example 1



(b) Example 2



(c) Example 3

Fig. 2. Consensus time courses for the examples of Fig. 1. While in Example 1 (structurally balanced $\mathcal{G}(A)$) the 3 agents converge to bipartite consensus, in Example 2 (structurally unbalanced) all 3 agents converge to 0 (i.e., no consensus is achieved). Example 3 is the gauge transformation of Example 1 in which all 3 agents have moved to the same side of the bipartition: in this case the problem becomes a standard consensus problem on a nonnegative weighted graph.

B. Directed graphs

Given a digraph $\mathcal{G}(A)$, we follow the convention of the consensus literature and call (row) Laplacian of A the matrix $L = C_r - A$. When A is digon sign-symmetric then we define $A_u = (A + A^T)/2$ as symmetrized adjacency matrix of the underlying undirected graph. Notice that in general $L_u = (L + L^T)/2 = C_r - A_u$ is different from $\hat{L}_u = C_u - A_u$ where

$C_u = (C_r + C_c)/2$. $L_u = \hat{L}_u$ if and only if $\mathcal{G}(A)$ is weight balanced.

Lemma 2 *A strongly connected, digon sign-symmetric signed digraph $\mathcal{G}(A)$ is structurally balanced if and only if any of the following equivalent conditions holds:*

1) *$\mathcal{G}(A_u)$ is structurally balanced;*
2) *all directed cycles of $\mathcal{G}(A)$ are positive;*
3) *$\exists\, D \in \mathcal{D}$ such that DAD has all nonnegative entries;*
4) *0 is an eigenvalue of L.*

Proof: From the digon sign-symmetry, $a_{ij}a_{ji} \geqslant 0$, which implies that for each entry of A_u $\mathrm{sgn}(a_{u,ij}) = \mathrm{sgn}(a_{ij})$ if $a_{ij} \neq 0$ and $\mathrm{sgn}(a_{u,ij}) = \mathrm{sgn}(a_{ji})$ if $a_{ji} \neq 0$ (or both, if $a_{ij},\, a_{ji} \neq 0$). Digon sign-symmetry implies also that the signs of the semicycles are the signs of the cycles of $\mathcal{G}(A_u)$. From Proposition 1 this implies that $\mathcal{G}(A)$ cannot have any negative directed cycle and viceversa, since otherwise $\mathcal{G}(A_u)$ cannot be structurally balanced. The third implication follows consequently from Lemma 1. As for the fourth condition, one direction is obvious: if A is structurally balanced then $\exists\, D \in \mathcal{D}$ such that DAD has all nonnegative entries, and we are in the usual consensus setting for nonnegative networks. To prove the converse, assume $\lambda_1(L) = 0$ is an eigenvalue of L. By construction, $\ell_{ii} = \sum_{j\neq i} |a_{ij}|$, which means that $\lambda_1(L) = 0$ is on the boundary of all the Geršgorin disks

$$\left\{ z \in \mathbb{C} \text{ s.t. } |z - \ell_{ii}| \leqslant \sum_{j\neq i} |a_{ij}| = \ell_{ii} \right\}. \tag{10}$$

Then from Lemma 6.2.3 of [23], since L is irreducible, it follows that the right eigenvector of 0, i.e., $w \neq 0$ for which $Lw = 0$, is such that $|w_i| = |w_j|\ \forall\, i, j = 1, \ldots, n$. We also have $w^T L^T = 0$ and hence $w^T(L + L^T)w = 0$. Since $L_u = (C_r - C_c)/2 + \hat{L}_u$, then

$$\frac{1}{2} w^T(L + L^T)w = w^T \left(\frac{C_r - C_c}{2} \right) w + w^T \hat{L}_u w = 0. \tag{11}$$

For the first term of (11), denoting $\omega = |w_i|\ \forall\, i = 1, \ldots, n$,

$$\begin{aligned} & w^T (C_r - C_c)\, w = \mathrm{tr}(C_r - C_c)\omega^2 \\ & = \left(\sum_i \sum_{j\neq i} |a_{ij}| - \sum_i \sum_{j\neq i} |a_{ji}| \right) \omega^2 = 0 \qquad \forall\, \omega. \end{aligned}$$

As for the second term of (11), it represents the Laplacian potential (computed in w) of an undirected graph, hence as in (3) it is in the form of a sum of squares. Assume now by contradiction that A has a negative semicycle. This implies that also A_u has to have a negative undirected cycle. The proof by contradiction now carries over from Lemma 1. ∎

Notice that if $\mathcal{G}(A)$ is weight balanced then, since A is structurally balanced iff A_u is, the last statement follows also from the well-known inequality (see e.g. [23], p. 187)

$$\min \mathrm{sp}(L_u) \leqslant \mathrm{Re}(\mathrm{sp}(L)) \leqslant \max \mathrm{sp}(L_u).$$

In fact, $\min \mathrm{sp}(L_u) = 0$ iff A is structurally balanced. If not, $\min \mathrm{sp}(L_u) > 0$, hence $\min \mathrm{Re}(\mathrm{sp}(L)) > 0$.

Corollary 3 *A strongly connected, digon sign-symmetric signed digraph $\mathcal{G}(A)$ is structurally unbalanced if and only if any of the following holds:*

1) *$\mathcal{G}(A_u)$ is structurally unbalanced;*
2) *$\mathcal{G}(A)$ has at least one negative directed cycle;*
3) *$\nexists\, D \in \mathcal{D}$ rendering DAD nonnegative;*
4) *$\lambda_1(L) > 0$, i.e., $-L$ is Hurwitz.*

Proof: The first three statements follow straightforwardly from Lemma 2. As for the fourth, L is diagonally dominant (see Sect. III-B1 for a detailed definition), hence from the Geršgorin disk theorem the eigenvalues of L are located in the union of the disks (10). Then $\mathrm{Re}(\mathrm{sp}(L)) \geqslant 0$, and, from Lemma 2, $\mathrm{Re}(\mathrm{sp}(L)) > 0$ if and only if A is structurally unbalanced. ∎

It also follows from Lemma 2 that for strongly connected digraphs $\mathrm{rank}(L) = n - 1$ if and only if A is structurally balanced. In particular, any acyclic digraph is structurally balanced, hence these cases can be treated analogously to their nonnegative weight counterparts (i.e., rooted trees admit a bipartite consensus [40]).

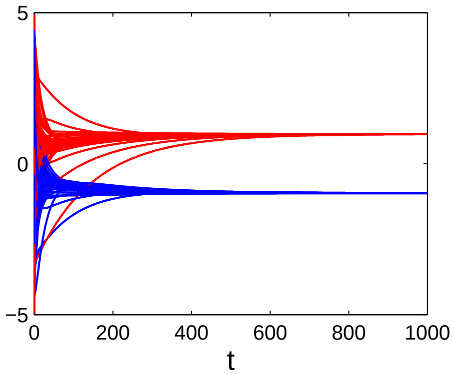
When $\mathcal{G}(A)$ is structurally balanced, denote ν a nonzero left eigenvector of DLD normalized such that $\nu^T \mathbf{1} = 1$, where $D \in \mathcal{D}$ s.t. DAD is nonnegative. We can now state the analogous of Thm. 1 for directed graphs.

Theorem 2 *Consider a strongly connected, digon sign-symmetric signed digraph $\mathcal{G}(A)$. The system* (4) *admits a bipartite consensus solution if and only if $\mathcal{G}(A)$ is structurally balanced. In this case $\lim_{t\to\infty} x(t) = \nu^T D x(0) D\mathbf{1}$, where $D \in \mathcal{D}$ is the gauge transformation such that DAD nonnegative. When $\mathcal{G}(A)$ is weight balanced $\lim_{t\to\infty} x(t) = \frac{1}{n}\left(\mathbf{1}^T D x(0)\right) D\mathbf{1}$. If instead $\mathcal{G}(A)$ is structurally unbalanced then $\lim_{t\to\infty} x(t) = 0\ \forall\, x(0) \in \mathbb{R}^n$.*

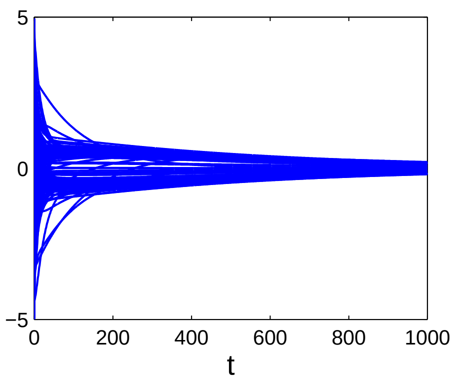
Proof: The first part follows from Lemma 2. The second from the observation that once we gauge transform the system via $z = Dx$ we have a standard consensus problem on a nonnegative directed graph. The final part instead follows from Corollary 3. ∎

From Lemma 2, the (nontrivial) bipartite consensus solution exists if and only if all directed cycles (or semicycles) of $\mathcal{G}(A)$ have positive sign, which is true if and only if $\lambda_1(L) = 0$. All these conditions are verifiable in polynomial time, meaning that verifying structural balance is an easy computational problem even in large-scale signed graphs. See [20] for an example of algorithm computing explicitly the bipartition.

Example 4 Fig. 3 shows in (a) the bipartite consensus achieved on a strongly connected structurally balanced signed digraph $\mathcal{G}(A)$ of $n = 1000$ agents. As soon as the sign is changed even on a single edge of $\mathcal{G}(A)$, structural balance is lost, and the agreement subspace becomes empty. From Theorem 2, $\lim_{t\to\infty} x(t) = 0\ \forall\, x(0) \in \mathbb{R}^n$, although the convergence rate can be very slow, see Fig. 3(b).



(a) Structurally balanced $\mathcal{G}(A)$



(b) Structurally unbalanced $\mathcal{G}(A)$

Fig. 3. Bipartite consensus on a strongly connected signed digraph with $n = 1000$. In (a) $\mathcal{G}(A)$ is structurally balanced, hence from Theorem 2 the agents split into two groups (in red and in blue). In (b) instead, a few edges have changed sign, unbalancing the graph. Bipartite consensus is now lost, and all agents converge (slowly) to 0.

1) A diagonal stability characterization: Quite remarkably, the nonsingularity of L for $\mathcal{G}(A)$ structurally unbalanced does not emerge from any of the standard linear-algebraic arguments based e.g. on Geršgorin disks theorem and/or on diagonal dominance. A matrix L is said *diagonally dominant* (by rows, omitted hereafter) if

$$|\ell_{ii}| \geqslant \sum_{j\neq i} |\ell_{ij}|, \quad i = 1, \dots, n. \tag{12}$$

It is said strictly diagonally dominant when the above inequalities are all strict and weakly diagonally dominant when at least one (but not all) of the inequalities (12) is strict. It is said *diagonally equipotent* if in (12) we have equality $\forall\, i = 1, \dots, n$.

As for the Geršgorin disk theorem, it affirms that the eigenvalues of L are located in the union of the n disks of (10). Diagonal dominance on the contrary guarantees that 0 cannot be in the interior of any of the Geršgorin disks, see [23], § 6.2. Hence, from (10), $z = 0$ is always on the boundary of all the disks regardless of structural balance. In fact, the Geršgorin disks depend on the diagonal values of L and on the absolute values of the off-diagonal entries of A. Therefore they cannot discern properties depending on the signs of the a_{ij}.

Example 5 For A_1 and A_2 of Examples 1 and 2, the inequalities (10) are identical, and so are the Geršgorin disks. However 0 is an eigenvalue of L_2 but not of L_1.

Similar considerations hold for the Cassini ovals and for all other known inclusion regions generalizing the Geršgorin disks (including Brualdi's cycles [9]).

All stability results for diagonally dominant matrices [25], [22], [30] are concerned with the strict/weak diagonally dominant case. Each time one of the inequalities in (10) is strict (meaning strict diagonal dominance on the corresponding row), 0 must be outside the corresponding Geršgorin disk. When L is diagonally equipotent, however, none of the sufficient conditions available in the literature apply.

The matrix $-L$ is said *diagonally stable* if $\exists$ a diagonal matrix $P = \mathrm{diag}(p_1, \dots, p_n)$, $p_i > 0$, s.t. $-PL - L^T P < 0$. The following Proposition will be useful later on.

Proposition 3 *Consider a strongly connected, digon sign-symmetric signed digraph $\mathcal{G}(A)$. Assume $\mathcal{G}(A)$ is weight balanced. The Laplacian matrix $-L$ is diagonally stable if and only if $\mathcal{G}(A)$ is structurally unbalanced.*

Proof: In the weight balanced case, we have

$$L_u = \hat{L}_u = (L + L^T)/2. \tag{13}$$

From condition 1 of Corollary 3, $\mathcal{G}(A)$ is structurally unbalanced iff $\mathcal{G}(A_u)$ is, hence $-L$ is Hurwitz iff $-\hat{L}_u$ is. But then (13) implies that $-L$ is diagonally stable with diagonal matrix $P = I$. Since $-L$ is Hurwitz iff $\mathcal{G}(A)$ is structurally unbalanced, the latter is also a necessary condition for diagonal stability. ∎

Remark 2 It is straightforward to check that when $\mathcal{G}(A)$ is weight balanced but structurally unbalanced $V = \|x\|^2$ is a Lyapunov function, since $\dot{V} = x^T(L + L^T)x = x^T\hat{L}_u x < 0$ $\forall\, x \in \mathbb{R}^n$. When instead $\mathcal{G}(A)$ is structurally balanced $\dot{V} = x^T\hat{L}_u x \leqslant 0$, as the agreement subspace $\ker(D\mathbf{1})$ is nontrivial, just like in the case of nonnegative weights, see [38].

2) Bipartite consensus under switching topologies: Consider r topologies on $\mathcal{V}$ defined by the signed adjacency matrices $A_1, \dots, A_r$ such that $\mathcal{G}(A_p)$ is digon sign-symmetric and strongly connected $\forall\, p \in \{1, \dots, r\}$. Assume further that the signs of the edges are never conflicting across the r digraphs:

$$\mathrm{sgn}(A_{p,ij})\mathrm{sgn}(A_{q,ij}) \geqslant 0 \qquad \begin{array}{l} \forall p, q \in \{1, \dots, r\}, \\ \forall i, j \in \{1, \dots, n\}, \quad i \neq j. \end{array} \tag{14}$$

Under these assumptions $\mathcal{G}(A_1), \dots, \mathcal{G}(A_r)$ can be rendered simultaneously nonnegative by the same gauge transformation $D \in \mathcal{D}$. Thm 9 of [38] holds for the family and assures convergence to $\frac{1}{n}\left(\mathbf{1}^T D x(0)\right) D\mathbf{1}$ for any switching pattern of the $\{A_1, \dots, A_r\}$.

Remark 3 Bipartite consensus under switching cannot be relaxed to gauge equivalent digraphs not obeying (14).

Example 6 A bipartite consensus cannot be achieved for the family $\{A_1, A_3\}$ of Examples 1 and 3 under arbitrary switching, since the corresponding steady states are different.

Remark 4 If $\mathcal{G}(A_p)$, $p \in \{1, \ldots, r\}$ obeying (14) are strongly connected but structurally unbalanced, then the entire polytope formed by the corresponding Laplacians $-L_1, \ldots, -L_r$ is Hurwitz. This follows from the diagonal stability of $-L_1, \ldots, -L_r$, with common Lyapunov function $V = \|x\|^2$, see Proposition 3 and Remark 2.

IV. Nonlinear consensus protocols

In this Section we investigate two families of nonlinear Laplacian consensus protocols. The first family is a generalization of the nonlinear schemes currently found in the literature for graphs of nonnegative weights [3], [27], [26], [34], [39], and it is based on summing up at each node the contribution of "local" nonlinearities (i.e., each feedback term of the summation relies only on the state of the node and of one of its neighbors). The second family of nonlinear consensus protocols is based instead on the notion of monotone dynamical systems. The formal analogy between the graphical tests for structural balance and monotonicity is exploited to design Laplacian feedback laws in which each term of the feedback may contain the states of all neighbors of a node.

A. Nonlinear distributed additive Laplacian feedback schemes

In the following we introduce first the structural properties of the class of systems used for the feedback (Section IV-A1) and then (Section IV-A2) devise their Laplacian counterpart.

1) Distributed nonlinear additive systems: Consider the system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{15}$$

where $f(x) = \begin{bmatrix} f_1(x) & \ldots & f_n(x) \end{bmatrix}^T$ is Lipschitz continuous, $f(0) = 0$. The following special forms for (15) are compatible with consensus-related problems.

(I) *Distributed vector fields:* $f(x)$ is distributed over the graph G, i.e., only the first neighbors of each agent matter in (15):

$$f_i(x) = f_i(x_j, \, j \in \mathrm{adj}(i)). \tag{16}$$

(II) *Distributed additive vector fields:* in (16) the contribution of each neighbor is additive

$$f_i(x) = \sum_{j \in \mathrm{adj}(i)} a_{ij} h_{ij}(x_j), \tag{17}$$

with $a_{ij} \in \mathbb{R}$ the weight of each contribution. Two special cases of (17) are listed in the following.

a) *All equal functions:*

$$h_{ij}(\cdot) = h(\cdot) \;\; \forall \, i, \, j = 1, \ldots, n,$$

meaning that we can write compactly (15) as

$$f(x) = A \begin{bmatrix} h(x_1) & \ldots & h(x_n) \end{bmatrix}^T = A h(x).$$

b) *All equal and antisymmetric functions:*

$$\begin{aligned} h_{ij}(x) = h(x) \;\; \forall \, i, \, j = 1, \ldots, n, \\ \text{and} \quad h(-x) = -h(x). \end{aligned} \tag{18}$$

In the following for these special cases we shall choose our $h(\cdot)$ in the class of (translated) positive, infinite sector nonlinearities $\mathcal{S}$, defined as follows:

$$\mathcal{S} = \Big\{ h \,:\, \mathbb{R} \to \mathbb{R}, \, (h(\xi) - h(\xi^*))\,(\xi - \xi^*) > 0 \text{ if } \xi \neq \xi^*,$$

$$h(0) = 0, \text{ and } \int_{\xi^*}^{\xi} (h(\tau) - h(\xi^*)) d\tau \to \infty \text{ as } |\xi - \xi^*| \to \infty \Big\}$$

Examples of functions in $\mathcal{S}$ are (translated) odd polynomials, tanh, etc. A subclass of $\mathcal{S}$ in which $\xi^* = 0$ is the following

$$\mathcal{S}_o = \Big\{ h \,:\, \mathbb{R} \to \mathbb{R}, \, h(\xi)\xi > 0 \text{ if } \xi \neq 0, \; h(0) = 0,$$

$$\text{and } \int_0^{\xi} h(\tau) d\tau \to \infty \text{ as } |\xi| \to \infty \Big\}.$$

When $h \in \mathcal{S}_o$, the Case IIa is a special situation of what is known in the literature as Persidskii systems [30].

2) Laplacian feedback schemes: For the system of integrators (2), various types of nonlinear additive Laplacian feedback schemes have been proposed in the literature [3], [27], [26], [34], [39] for the case of nonnegative weights in the adjacency matrix A. We shall consider two of them (the nomenclature given here follows [45]), extending them to any A:

- absolute Laplacian flow

$$\dot{x}_i = - \sum_{j \in \mathrm{adj}(i)} |a_{ij}| \left(h_{ij}(x_i) - \mathrm{sgn}(a_{ij}) h_{ij}(x_j) \right); \tag{19}$$

- relative Laplacian flow

$$\dot{x}_i = - \sum_{j \in \mathrm{adj}(i)} |a_{ij}| h_{ij}(x_i - \mathrm{sgn}(a_{ij}) x_j). \tag{20}$$

In Case IIa the absolute Laplacian flow (19) can be written as

$$\dot{x} = -L h(x), \tag{21}$$

where, as before, $L = C_r - A$. For the sake of simplicity we shall treat only the weight balanced case, although the arguments can be extended to the case of $\mathcal{G}(A)$ with a rooted tree (following the lines of e.g. [34]).

Theorem 3 *Consider a strongly connected, digon sign-symmetric, weight balanced, signed digraph $\mathcal{G}(A)$. The system* (21) *with $h \in \mathcal{S}$ admits a bipartite consensus if and only if $\mathcal{G}(A)$ is structurally balanced. In this case* $\lim_{t \to \infty} x(t) = \frac{1}{n} \left(\mathbf{1}^T D x(0) \right) D \mathbf{1}$, *where $D \in \mathcal{D}$ is the gauge transformation such that DAD nonnegative.*

Proof: From our previous analysis, $-L$ is Hurwitz stable when $\mathcal{G}(A)$ is structurally unbalanced, and critically stable when $\mathcal{G}(A)$ structurally balanced. For $h \in \mathcal{S}$, depending on $\mathcal{G}(A)$, the equilibrium point of (21) is

- x^* such that $h(x^*) = \begin{bmatrix} h(x_1^*) \ldots h(x_n^*) \end{bmatrix}^T = \alpha D \mathbf{1}$ for some nonzero α when $\mathcal{G}(A)$ is structurally balanced ($D \in \mathcal{D}$ s.t. DAD nonnegative);
- $x^* = 0$ when $\mathcal{G}(A)$ structurally unbalanced.

In the structurally balanced case, whenever $h \in \mathcal{S}$, the diagonal form of the nonlinearity implies that it must be $|x_i^*| = |x_j^*|$, and in particular $x^* = \beta D\mathbf{1}$ for some $\beta \neq 0$. Applying the change of coordinates $z = Dx$ we obtain $\dot{z} = -DLh(Dz)$ with

$$\mathrm{sgn}(h(\sigma_i z_i)) = \mathrm{sgn}(\sigma_i h(z_i)) = \sigma_i \mathrm{sgn}(h(z_i)). \tag{22}$$

Therefore, from Lemma 2, $\mathcal{G}(A)$ structurally balanced means that $\exists\, D \in \mathcal{D}$ such that all off-diagonal elements $\sigma_i a_{ij} \sigma_j$ are nonnegative in the z-coordinates. Consider the following integral Lyapunov function:

$$\begin{aligned} V(x) &= \sum_i \int_{x_i^*}^{x_i} \left(h(\xi) - h(x_i^*)\right) d\xi \qquad h \in \mathcal{S} \\ &= \sum_i \int_{\beta\sigma_i}^{x_i} \left(h(\xi) - \alpha\sigma_i\right) d\xi. \end{aligned} \tag{23}$$

From (22) and $\sigma_i^2 = 1$, the sign of each term in (23) is invariant to gauge transformations. By construction, $V(x) > 0$, $V(x^*) = 0$ and $V(x)$ radially unbounded. Its derivative is

$$\begin{aligned} \dot{V}(x) &= \sum_i \left(h(x_i) - h(x_i^*)\right) \dot{x}_i \\ &= -\left(h(x) - h(x^*)\right)^T L h(x) \\ &= -\frac{1}{2} h(x)^T L h(x), \end{aligned}$$

since $h(x^*) = \alpha D\mathbf{1}$ and $D\mathbf{1}$ is a left eigenvector of L when $\mathcal{G}(A)$ is weight balanced. Only the symmetric part of L matters in a quadratic form, therefore

$$\dot{V}(x) = -\frac{1}{2} h(x)^T (L + L^T) h(x) = h(x)^T L_u h(x) \leqslant 0,$$

since $L_u = \hat{L}_u = (L + L^T)/2$ is positive semidefinite, with $\ker(\hat{L}_u) = \mathrm{span}(D\mathbf{1})$. When $\mathcal{G}(A)$ is weight balanced, integrating the conservation law $-\mathbf{1}^T DLh(x) = \mathbf{1}^T D\dot{x} = 0$ implies

$$\mathbf{1}^T Dx(t) = \mathbf{1}^T Dx(0) \ \ \forall t > 0,$$

i.e., $\lim_{t\to\infty} x(t) = \frac{1}{n}\left(\mathbf{1}^T Dx(0)\right) D\mathbf{1}$. When instead $\mathcal{G}(A)$ is structurally unbalanced, then $x^* = 0$, $h(x^*) = 0$, i.e., $h \in \mathcal{S}_o$. From Proposition 3, $-L$ is diagonally stable with I as diagonal matrix. It follows that $\dot{V}(x) < 0$, meaning that $x^* = 0$ is globally asymptotically stable, i.e., only trivial consensus is achieved. ∎

Also for the relative Laplacian (20) a similar result holds whenever the feedback functions h_{ij} are all equal and anti-symmetric (i.e., in Case IIb).

Theorem 4 *Consider a strongly connected, digon sign-symmetric, weight balanced, signed digraph $\mathcal{G}(A)$. The system* (20) *with h_{ij} obeying* (18) *and $h \in \mathcal{S}_o$ admits a bipartite consensus if and only if $\mathcal{G}(A)$ is structurally balanced. In this case $\lim_{t\to\infty} x(t) = \frac{1}{n}\left(\mathbf{1}^T Dx(0)\right) D\mathbf{1}$, where $D \in \mathcal{D}$ is the gauge transformation such that DAD nonnegative.*

Proof: Assume $\mathcal{G}(A)$ structurally balanced. If $D \in \mathcal{D}$ is the gauge transformation such that DAD is nonnegative, then in the z coordinates (20) becomes

$$\begin{aligned} \dot{z}_i &= -\sigma_i \sum_{j\in \mathrm{adj}(i)} |a_{ij}| h(\sigma_i z_i - \mathrm{sgn}(a_{ij})\sigma_j z_j) &(24)\\ &= -\sigma_i \sum_{j\in \mathrm{adj}(i)} |a_{ij}| h\left(\sigma_i(z_i - \sigma_i \mathrm{sgn}(a_{ij})\sigma_j z_j)\right) &(25)\\ &= -\sum_{j\in \mathrm{adj}(i)} |a_{ij}| h(z_i - z_j), &(26) \end{aligned}$$

because $\sigma_i \mathrm{sgn}(a_{ij})\sigma_j \geqslant 0$ by construction and, when $\sigma_i = -1$, (18) holds. This is now a standard consensus problem on a nonnegative weight balanced digraph. A possible proof, valid in the case in which A is non-symmetric is provided in [34], based on a min-max type of Lyapunov function. Another approach to the same problem is discussed in [27]. Clearly in a structurally balanced case any design valid for a nonnegative weight balanced system (26), is valid also for the original signed graph.

When instead $\mathcal{G}(A)$ is not structurally balanced, then from Lemma 2 not all negative weights of $\mathcal{G}(A)$ can be simultaneously eliminated by any $D \in \mathcal{D}$. Assuming by contradiction that a nontrivial bipartite consensus exists, then from Definition 2 it must correspond to a nonzero equilibrium x^* such that $|x_i^*| = |x_j^*| \ \forall\, i, j = 1, \ldots, n$. This implies $x^* = \alpha D\mathbf{1}$ for some $D \in \mathcal{D}$ and some $\alpha > 0$. It is enough to show that no such equilibrium point x^* can exist for (20) when $\alpha > 0$. In fact, from $x^* = \alpha D\mathbf{1}$ we have that applying the gauge transformation $z = Dx$, in the z-coordinates (20) becomes, analogously to (24)-(25),

$$\dot{z}_i = -\sum_{j\in \mathrm{adj}(i)} |a_{ij}| h(z_i - \mathrm{sgn}(\sigma_i a_{ij} \sigma_j) z_j), \tag{27}$$

where we have applied (18). By construction, in the z-coordinates $z^* = \alpha\mathbf{1}$, i.e., $z_i^* > 0 \ \forall\, i = 1, \ldots, n$. From Lemma 2, in (27) not all $\sigma_i a_{ij} \sigma_j$ can be nonnegative, unlike in (26). Hence at z^* in (27) we must necessarily have $z_i^* - \mathrm{sgn}(\sigma_i a_{ij} \sigma_j) z_j^* = z_i^* + z_j^* = 2\alpha > 0$ for at least a pair (i, j), while, whenever $\mathrm{sgn}(\sigma_i a_{ij} \sigma_j) > 0$, $z_i^* - \mathrm{sgn}(\sigma_i a_{ij} \sigma_j) z_j^* = z_i^* - z_j^* = 0$. Since $h \in \mathcal{S}_o$, $h(\xi) > 0$ when $\xi > 0$, hence at least one of the $\dot{z}_i$ must be strictly negative and z^* cannot be an equilibrium point. ∎

B. Nonlinear Laplacian flow with monotone laws

In this final Section of the paper we first show that structural balance and monotonicity of a dynamical system have identical graphical tests, then present a distributed nonlinear feedback method to obtain bipartite consensus starting from a monotone system.

1) Monotone systems: In $\mathbb{R}^n$, consider the orthant of $\mathbb{R}^n$ corresponding to σ: $K_\sigma = \{x \in \mathbb{R}^n$ such that $Dx \geqslant 0, D \in \mathcal{D}\}$, and denote by $\phi^t(x_1)$ the solution of (15) at time t in correspondence of the initial condition x_1. The partial order generated by σ is normally indicated by the symbol "$\leqslant_\sigma$": $x_1 \leqslant_\sigma x_2 \iff x_2 - x_1 \in K_\sigma$. Strict ordering is denoted $x_1 <_\sigma x_2$ and corresponds to $x_1 \leqslant_\sigma x_2$, $x_1 \neq x_2$. When inequality must hold for all coordinates of x_1, x_2 then we use the notation "$\ll_\sigma$".

Definition 3 *The system* (15) *is said* monotone *with respect to the partial order* σ *if for all initial conditions* x_1, x_2 *such that* $x_1 \leqslant_\sigma x_2$ *one has* $\phi^t(x_1) \leqslant_\sigma \phi^t(x_2)$ $\forall\, t \geqslant 0$. *It is said* strongly monotone *with respect to the partial order* σ *if for all initial conditions* x_1, x_2 *such that* $x_1 <_\sigma x_2$ *one has* $\phi^t(x_1) \ll_\sigma \phi^t(x_2)$ $\forall\, t > 0$.

Denote $F(x) = \frac{\partial f(x)}{\partial x}$ the Jacobian of (15) at x, of elements $F_{ij}(x)$. Assume $F(x)$ is *sign constant*: $F_{ij}(x_1)F_{ij}(x_2) \geqslant 0$ $\forall\, x_1, x_2 \in \mathbb{R}^n$, $\forall\, i, j \in \{1, \ldots, n\}$, and *sign-symmetric*: $F_{ij}(x)F_{ji}(x) \geqslant 0$ $\forall\, x \in \mathbb{R}^n$. See [42], [43] for more details on monotone systems.

Denote $\mathcal{G}(F(x))$ the graph of which $F(x)$ is the adjacency matrix at $x \in \mathbb{R}^n$. We say that $\mathcal{G}(F(x))$ is *globally strongly connected* if $\mathcal{G}(F(x))$ strongly connected $\forall\, x \in \mathbb{R}^n$. This is equivalent to have $F(x)$ irreducible $\forall\, x \in \mathbb{R}^n$. Given $\epsilon > 0$, we say further that $\mathcal{G}(F(x))$ is *globally ϵ-strongly connected* if $\mathcal{G}(\hat{F}(x))$ is globally strongly connected, where $\hat{F}(x)$ is such that

$$\hat{F}_{ij}(x) = \begin{cases} F_{ij}(x) & \text{if } |F_{ij}(x)| \geqslant \epsilon \\ 0 & \text{if } |F_{ij}(x)| < \epsilon. \end{cases}$$

In the spirit of [37] (but in a much stronger sense), this property guarantees that the graph remains strongly connected even when too small edge weights (below the threshold ϵ) are disregarded.

Let us now evaluate the effect of a gauge transformation $D = \mathrm{diag}(\sigma)$ (i.e., of the change of partial order σ) on a strongly monotone system. As before, consider the change of coordinates $z = Dx$. Since $F_{ij}(x)$ is sign constant $\forall\, x \in \mathbb{R}^n$, it must be that also $F_{ij}(z)$ is sign constant $\forall\, z \in \mathbb{R}^n$ and in addition $\mathrm{sgn}(F_{ij}(x))\mathrm{sgn}(F_{ij}(z)) \geqslant 0$ $\forall\, x \in \mathbb{R}^n$. From $D^{-1} = D$, the new Jacobian is $F_D(z) = DF(z)D$.

Lemma 3 *Consider a system* (15) *whose Jacobian is sign constant and sign-symmetric. Then* (15) *is strongly monotone in* $\mathbb{R}^n$ *if and only if any of the following conditions holds:*

1) $F(x)$ *is irreducible* $\forall\, x \in \mathbb{R}^n$ *and* $\exists\, D \in \mathcal{D}$ *such that* $DF(z)D$ *has all nonnegative entries* $\forall\, x \in \mathbb{R}^n$*;*
2) $\mathcal{G}(F(x))$ *is globally strongly connected and structurally balanced* $\forall\, x \in \mathbb{R}^n$*;*
3) $\mathcal{G}(F(x))$ *is globally strongly connected and all directed cycles of* $\mathcal{G}(F(x))$ *are positive* $\forall\, x \in \mathbb{R}^n$.

Proof: The first condition is the so-called Kamke lemma [43]. Together with irreducibility of $F(x)$ $\forall\, x \in \mathbb{R}^n$ (i.e., global strong connectivity of $\mathcal{G}(F(x))$) it corresponds to strong monotonicity in $\mathbb{R}^n$. If $F(x)$ is the adjacency matrix of a signed digraph at x, then $\mathcal{G}(F(x))$ is digon sign-symmetric and strongly connected by construction $\forall\, x \in \mathbb{R}^n$. This implies that Lemma 2 holds $\forall\, x \in \mathbb{R}^n$. The equivalence between condition 1 and conditions 2 and 3 follows consequently. ■

2) Nonlinear Laplacian flow with monotone laws: Consider a strongly monotone distributed system of the form (16). Since $F_{ij}(x) \neq 0$ implies $j \in \mathrm{adj}(i)$, also the Jacobian linearization at each $x \in \mathbb{R}^n$ is distributed. It is also sign constant by assumption. Hence $F(x)$ can be used to obtain the following nonlinear Laplacian scheme:

$$\begin{aligned} \dot{x}_i &= -\sum_{j \in \mathrm{adj}(i)} \left(|F_{ij}(x)| x_i - F_{ij}(x) x_j\right) \\ &= -\sum_{j \in \mathrm{adj}(i)} |F_{ij}(x)| \left(x_i - \mathrm{sgn}(F_{ij}(x)) x_j\right), \end{aligned} \tag{28}$$

for which the following Theorem holds.

Theorem 5 *Any distributed strongly monotone system* (16) *whose Jacobian is globally ϵ-strongly connected admits a Laplacian flow* (28) *which is globally converging to a bipartite consensus.*

Proof: Under the gauge transformation $z = Dx$, (28) becomes

$$\dot{z}_i = -\sum_{j \in \mathrm{adj}(i)} \left(|F_{ij}(z)| z_i - \sigma_i \sigma_j F_{ij}(z) z_j\right). \tag{29}$$

Strong monotonicity of $f(x)$ implies that $F_{ij}(x)$ is sign constant $\forall\, x \in \mathbb{R}^n$ and that $\exists\, D \in \mathcal{D}$ such that $\sigma_i \sigma_j F_{ij}(z) \geqslant 0$. For this D, (29) is therefore

$$\dot{z}_i = -\sum_{j \in \mathrm{adj}(i)} |F_{ij}(z)| (z_i - z_j), \tag{30}$$

i.e., a nonlinear Laplacian scheme on a (state-dependent) nonnegative weighted graph. The assumption of global ϵ-strong connectivity guarantees convergence $\forall\, x \in \mathbb{R}^n$. ■

The conditions of Thm 5 are sufficient but not necessary. For example one can think of relaxing the strong connectivity in several ways, see e.g. [1], [11], [33], [35].

Example 7 Consider the system in Fig. 4(a) and the following vector field:

$$f(x) = \begin{bmatrix} \theta e^{\mu_2 x_2} e^{\mu_3 x_3} + a_{14} x_4 \\ a_{21} x_1 \\ a_{32} x_2 \\ a_{43} x_3 \end{bmatrix}$$

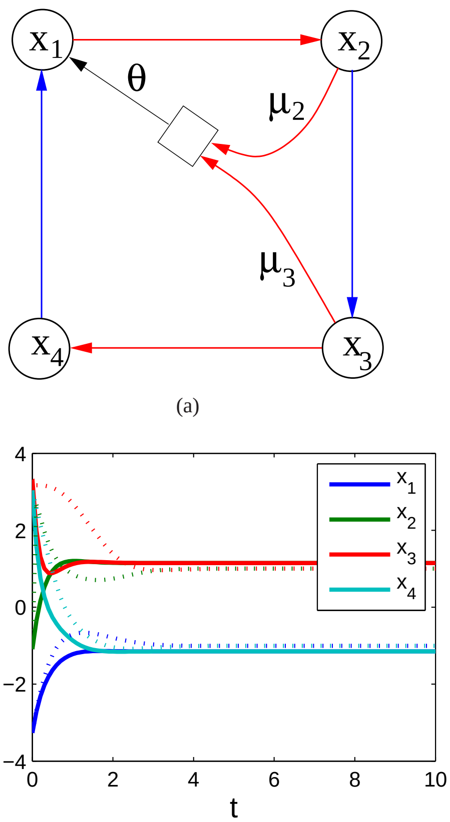
where the nonlinear (and "nonlocal") term is depicted as a square box in Fig. 4(a). The Jacobian of $f(x)$ is

$$F(x) = \begin{bmatrix} 0 & \theta\mu_2 \prod_{i=2,3} e^{\mu_i x_i} & \theta\mu_3 \prod_{i=2,3} e^{\mu_i x_i} & a_{14} \\ a_{21} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{43} & 0 \end{bmatrix}.$$

$F(x)$ is sign constant $\forall\, x \in \mathbb{R}^4$. When $|a_{ij}| > \epsilon$, clearly $\mathcal{G}(F(x))$ is also ϵ-strongly connected $\forall\, x \in \mathbb{R}^4$, hence when $f(x)$ is strongly monotone, i.e., when $\exists\, D \in \mathcal{D}$ such that $DF(x)D$ has all nonnegative entries, then Thm. 5 applies. In this case the nonlinear Laplacian feedback one obtains

$$\begin{aligned} \dot{x}_1 &= -\theta \prod_{i=2,3} e^{\mu_i x_i} \sum_{i=2,3} |\mu_i| \left(x_1 - \mathrm{sgn}(\mu_i) x_i\right) \\ &\quad -|a_{14}|(x_1 - \mathrm{sgn}(a_{14}) x_4) \\ \dot{x}_2 &= -|a_{21}|(x_2 - \mathrm{sgn}(a_{21}) x_1) \\ \dot{x}_3 &= -|a_{32}|(x_3 - \mathrm{sgn}(a_{32}) x_2) \\ \dot{x}_4 &= -|a_{43}|(x_4 - \mathrm{sgn}(a_{43}) x_3) \end{aligned}$$

globally converges to a bipartite consensus. For example, assuming θ, a_{14}, $a_{32} > 0$ and μ_1, μ_2, a_{21}, $a_{43} < 0$, $\mathcal{G}(F(x))$ is structurally balanced with $D = \mathrm{diag}(1,\,-1,\,-1,\,1)$. Simulations with (solid curves) and without (dotted) the nonlinear term are shown in Fig. 4(b). Clearly the presence of the nonlinearity speeds up the convergence rate. Notice that the lack of weight balance implies the value $|x^*|$ of the bipartite consensus can change.



(a)

(b)

Fig. 4. Connectivity graph of Example 7 and corresponding time courses. The dotted curves correspond to $\theta = 0$.

V. Conclusion and Outlook

This paper extends the notion of consensus and its distributed feedback designs to networks containing interactions which are competitive in nature, modeled as negative weights on the communication edges. In this broader scenario, the conditions under which a consensus is achievable are formally analogous to those that characterize monotonicity of a system. The consensus reached is bipartite, i.e., the agents agree to a common (absolute) value but polarize themselves in two opposite fronts.

Since a bipartite consensus problem on a structurally balanced signed network is equivalent, up to a gauge transformation, to a standard consensus problem on a nonnegative network, a number of properties of the latter are valid also for the former. These include several aspects not discussed in the paper, like finite-time semistability [26], consensus in presence of time-delays or when the strong connectivity assumption is relaxed. Less straightforward extensions include dealing with the discrete-time case and with higher order integrators.

From a system theory perspective, the case when bipartite consensus cannot be achieved is equally interesting: in fact the corresponding Laplacians (linear or nonlinear) induce a globally asymptotically stable closed loop system. In the linear case, for example, this form of global convergence is not classifiable in terms of standard linear-algebraic criteria, like diagonal dominance or location of the Geršgorin disks.

Appendix

A. Proof of Proposition 1

One implication is obvious, since directed cycles are a subset of semicycles. To prove the opposite implication, assume all directed cycles are positive and, by contradiction, that $\exists$ a negative semicycle $\mathcal{C} \subset \mathcal{E}$ which is not a directed cycle. Denote $k_{\mathcal{C}}$ the length of $\mathcal{C}$. $\mathcal{C}$ can be broken into the concatenation of k ($k \leqslant k_{\mathcal{C}}$) directed (simple) paths with alternating directions, call them $\overrightarrow{\mathcal{P}}_1, \overleftarrow{\mathcal{P}}_2, \overrightarrow{\mathcal{P}}_3, \ldots, \overleftarrow{\mathcal{P}}_k \subset \mathcal{E}$. Without loss of generality assume

$$\begin{aligned} &\mathrm{sgn}(\overrightarrow{\mathcal{P}}_1) < 0, \\ &\mathrm{sgn}(\overleftarrow{\mathcal{P}}_2) = \mathrm{sgn}(\overrightarrow{\mathcal{P}}_3) = \ldots = \mathrm{sgn}(\overleftarrow{\mathcal{P}}_k) > 0. \end{aligned} \tag{31}$$

Denote $\mathcal{V}_{\overrightarrow{\mathcal{P}}_1}, \mathcal{V}_{\overleftarrow{\mathcal{P}}_2}, \ldots, \mathcal{V}_{\overleftarrow{\mathcal{P}}_k}$ the node sets of $\overrightarrow{\mathcal{P}}_1, \overleftarrow{\mathcal{P}}_2, \ldots, \overleftarrow{\mathcal{P}}_k$. If $v_{1_1}, v_{1_p} \in \mathcal{V}_{\overrightarrow{\mathcal{P}}_1}$ are the root and terminal nodes of $\overrightarrow{\mathcal{P}}_1$, then by the strong connectivity assumption $\exists$ a path $\overleftarrow{\mathcal{P}}_1$ connecting v_{1_p} to v_{1_1}, and similarly for $\overleftarrow{\mathcal{P}}_2, \overrightarrow{\mathcal{P}}_3, \ldots, \overleftarrow{\mathcal{P}}_k$. Call $\overrightarrow{\mathcal{P}}_2, \overrightarrow{\mathcal{P}}_4, \ldots, \overrightarrow{\mathcal{P}}_k \subset \mathcal{E}$ the paths completing $\overleftarrow{\mathcal{P}}_2, \overleftarrow{\mathcal{P}}_4, \ldots, \overleftarrow{\mathcal{P}}_k$ to directed cycles. Assume for the moment that

$$\mathcal{V}_{\overrightarrow{\mathcal{P}}_i} \cap \mathcal{V}_{\overrightarrow{\mathcal{P}}_j} = \emptyset \;\forall\, i, j = 1, \ldots, k. \tag{32}$$

Since by assumption all directed cycles are positive,

$$\mathrm{sgn}(\overrightarrow{\mathcal{P}}_i) = \mathrm{sgn}(\overleftarrow{\mathcal{P}}_i) \quad \forall\, i = 2, 4 \ldots, k. \tag{33}$$

By concatenating $\overrightarrow{\mathcal{P}}_1$ with $\overrightarrow{\mathcal{P}}_2, \overrightarrow{\mathcal{P}}_3, \ldots \overrightarrow{\mathcal{P}}_k$, from (33) the directed cycle $\overrightarrow{\mathcal{C}} = \bigcup_{i=1}^{k} \overrightarrow{\mathcal{P}}_i$ must have the same sign as $\mathcal{C}$, which is a contradiction since all directed cycles must be positive.

When (32) is not valid, then $\overrightarrow{\mathcal{C}}$ is a union of directed cycles. Assume for example that two of the directed paths $\overrightarrow{\mathcal{P}}_i$ and $\overrightarrow{\mathcal{P}}_j$, $i < j$, share a common subpath: $\overrightarrow{\mathcal{P}}_i = \overrightarrow{\mathcal{Q}}_{i,1} \cup \overrightarrow{\mathcal{Q}} \cup \overrightarrow{\mathcal{Q}}_{i,2}$, $\overrightarrow{\mathcal{P}}_j = \overrightarrow{\mathcal{Q}}_{j,1} \cup \overrightarrow{\mathcal{Q}} \cup \overrightarrow{\mathcal{Q}}_{j,2}$. Then the cycle $\overrightarrow{\mathcal{C}}$ is not simple (i.e, some of its nodes have connectivity more than 2) and it splits into the two (simple) cycles:

$$\begin{aligned} \overrightarrow{\mathcal{C}}_1 = {} & \overrightarrow{\mathcal{P}}_1 \cup \ldots \cup \overrightarrow{\mathcal{P}}_{i-1} \cup \overrightarrow{\mathcal{Q}}_{i,1} \cup \overrightarrow{\mathcal{Q}} \cup \\ & \cup \overrightarrow{\mathcal{Q}}_{j,2} \cup \overrightarrow{\mathcal{P}}_{j+1} \cup \ldots \cup \overrightarrow{\mathcal{P}}_k, \\ \overrightarrow{\mathcal{C}}_2 = {} & \overrightarrow{\mathcal{Q}}_{i,2} \cup \overrightarrow{\mathcal{P}}_{i+1} \cup \ldots \cup \overrightarrow{\mathcal{P}}_{j-1} \cup \overrightarrow{\mathcal{Q}}_{j,1} \cup \overrightarrow{\mathcal{Q}}. \end{aligned}$$

From (31) and (33), $\mathrm{sgn}(\overrightarrow{\mathcal{P}}_i) = \mathrm{sgn}(\overrightarrow{\mathcal{P}}_j) > 0$ and

$$\mathrm{sgn}(\overrightarrow{\mathcal{P}}_\ell) = \mathrm{sgn}(\overrightarrow{\mathcal{Q}}_{\ell,1} \cup \overrightarrow{\mathcal{Q}}_{\ell,2})\mathrm{sgn}(\overrightarrow{\mathcal{Q}}), \quad \ell = i, j.$$

We have then that $\text{sgn}(\overrightarrow{\mathcal{Q}}_{i,1} \cup \overrightarrow{\mathcal{Q}}_{i,2}) = \text{sgn}(\overrightarrow{\mathcal{Q}}_{j,1} \cup \overrightarrow{\mathcal{Q}}_{j,2})$ is $+1$ if $\text{sgn}(\overrightarrow{\mathcal{Q}}) = +1$, -1 otherwise. In the first case, for example, it must be $\text{sgn}(\overrightarrow{\mathcal{Q}}_{\ell,1}) = \text{sgn}(\overrightarrow{\mathcal{Q}}_{\ell,2})$, $\ell = i, j$. If in addition $\text{sgn}(\overrightarrow{\mathcal{Q}}_{i,1}) = \text{sgn}(\overrightarrow{\mathcal{Q}}_{j,1})$ then $\text{sgn}(\overrightarrow{\mathcal{C}}_1) = -1$ and $\text{sgn}(\overrightarrow{\mathcal{C}}_2) = +1$. If instead $\text{sgn}(\overrightarrow{\mathcal{Q}}_{i,1}) = -\text{sgn}(\overrightarrow{\mathcal{Q}}_{j,1})$ then $\text{sgn}(\overrightarrow{\mathcal{C}}_1) = +1$ and $\text{sgn}(\overrightarrow{\mathcal{C}}_2) = -1$. The other case (i.e., $\text{sgn}(\overrightarrow{\mathcal{Q}}) = -1$) can be treated analogously. Each subpath $\overrightarrow{\mathcal{Q}}$ (or even just single node) shared by two or more of the paths $\overrightarrow{\mathcal{P}}_1, \ldots, \overrightarrow{\mathcal{P}}_k$ gives rise to a splitting of $\overrightarrow{\mathcal{C}}$ into directed cycles: $\overrightarrow{\mathcal{C}} = \overrightarrow{\mathcal{C}}_1 \cup \overrightarrow{\mathcal{C}}_2 \cup \ldots \cup \overrightarrow{\mathcal{C}}_\ell$ for some $\ell > 1$. Since $\text{sgn}(\overrightarrow{\mathcal{C}}) = \text{sgn}(\mathcal{C}) = -1$ still holds and $\text{sgn}(\overrightarrow{\mathcal{C}}) = \prod_i \text{sgn}(\overrightarrow{\mathcal{C}}_i)$, then necessarily at least one of the $\overrightarrow{\mathcal{C}}_i$ must have negative sign and we are still in contradiction.

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