DFT—THE DISCRETE FOURIER TRANSFORM

The *discrete Fourier transform* (*DFT*) is defined as

$$X(k) =^{\Delta} \sum_{n=0}^{N-1} x(n) W^{nk} \qquad , \; k = 0, 1, \ldots, N-1$$

and the *inverse discrete Fourier transform* (*IDFT*) is

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W^{-nk} \qquad , \; n = 0, 1, \ldots, N-1$$

where $W = e^{-j2\pi/N}$

Direct Computation of the DFT

```
Program Direct_DFT;
var
   x, Y: array[0..Nminus1] of Complex;
begin
   for k := 0 to N-1 do
      begin
         Y[k] := x[0];
         for n := 1 to N-1 do
         Y[k] := Y[k] + Wnk * x[n];
      end;
end.
```

FFT — THE FAST FOURIER TRANSFORM ALGORITHM

In 1965, Cooley and Tukey developed a fast algorithm based on the divide-and-conquer principle which requires only $O(N \log_2(N))$ complex operations.

The Cooley-Tukey FFT

Radix-2 FFT

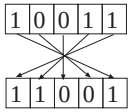
```
Program CT_FFT;
   const
      N = 1024;  M = 10;  Nminus1 = 1023; { N = 2^M }
   type
      Complex = record
         re : Double; im : Double;
      end;
      C_array : array[0..Nminus1] of Complex;
   var
      Stage, Ns, M1, k, kNs,  p, q : integer;
      WCos, WSin, TwoPiN, TempRe, TempIm : Double;
      x : C_array;

   begin  { READ INPUT DATA INTO x }
      Ns := N;  M1 := M;
      TwoPiN := 2 * Pi/N;
```



```
      for Stage := 1 to M do
         begin
            k := 0;
            Ns := Ns div 2;
            M1 := M1 - 1;
            while k < N do
            begin
            for q := 1 to Ns  do
            begin
            p := Digit_Reverse(k/2^M1);
            WCos := cos(TwoPiN * p); {W to the power of p }
            WSin := -sin(TwoPiN * p); {W = exp(-j2π/N) }
            kNs := k + Ns;
            TempRe := x[kNs].re * WCos - x[kNs].im * WSin;
            TempIm := x[kNs].im * WCos + x[kNs].re * WSin;
            x[kNs].re := x[k].re - TempRe;
            x[kNs].im := x[k].im - TempIm;
            x[k].re := x[k].re + TempRe;
            x[k].im := x[k].im + TempIm;
            k := k + 1;
            end;
            k := k + Ns;
            end;
         end;
         Unscramble; { OUTPUT DATA STORED IN x }
end.
```


Digit Reversal

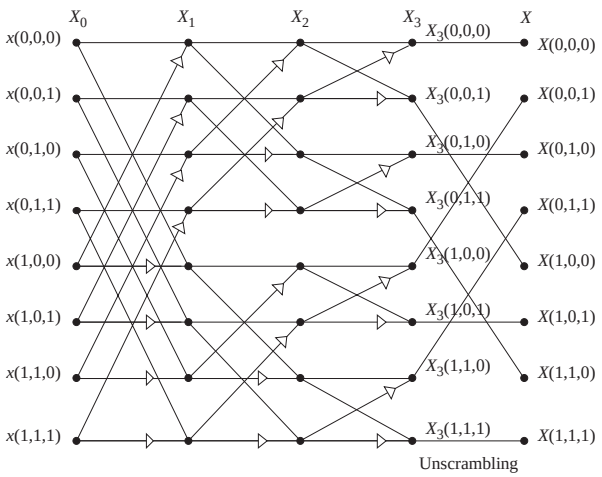
```
function Digit_Reverse(Digit: Integer) : Integer;
var
   N, q, NewAddr, Rmbit, OldAddr : Integer;
begin
   NewAddr := 0;
   OldAddr := Digit;
   for q := 1 to M do
      begin
      Rmbit := OldAddr mod 2;
      OldAddr := OldAddr div 2;
      if Rmbit = 1 then
         NewAddr := NewAddr * 2 + 1
      else
         NewAddr := NewAddr + NewAddr;
      end;
   Digit_Reverse := NewAddr;
end;
```



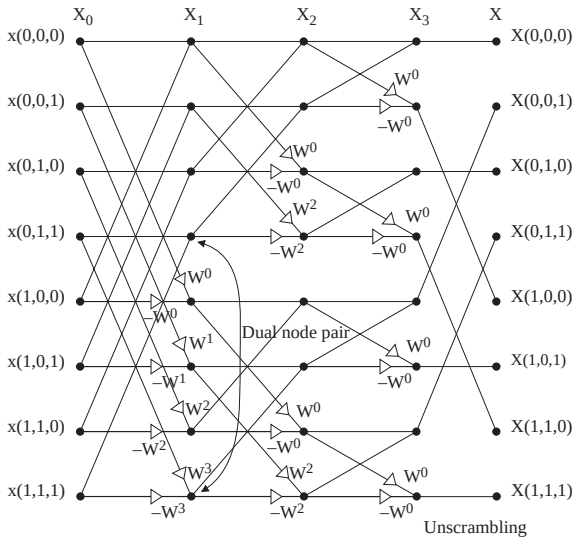
Unscramble

```
procedure Unscramble;
   var
      temp : Complex;
      k, q : integer;
   begin
      for k := 0 to N - 1 do
         begin
         q := Digit_Reverse(k);
         if q > k then
            begin
            temp := x[k];
            x[k] := x[q];
            x[q] := temp;
            end;
         end;
   end;
```

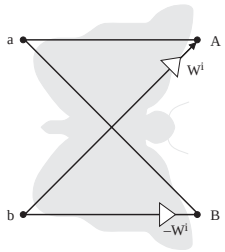

Cooley-Tukey FFT Signal-Flow Graph



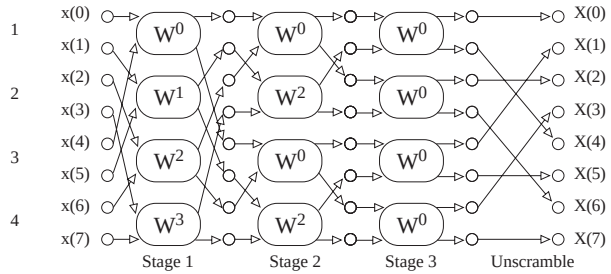
Sande-Tukey's FFT Signal-Flow Graph



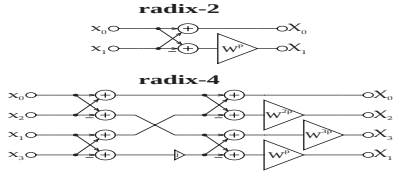
Radix-2 Butterfly



Butterfly op



Radix-4 Butterfly



Radix-8 Butterfly

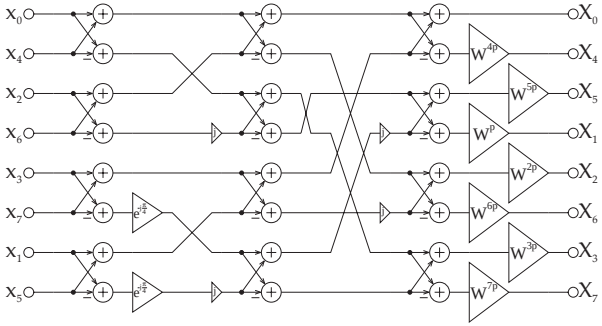


Table 1: Radix-*r* 4096-points FFT Algorithms

Operation	Radix-2	Radix-4	Radix-8
Complex Multiplications	22528	15360	10752
Real Multiplications	0	0	8192
Complex Additions	49152	49152	49152
Real Additions	0	0	8192
Memory Accesses	49152	24576	16384

The Inverse FFT

The inverse FFT (*IFFT*) can be computed by a simple modification of the input read process.

The IFFT is obtained by computing the FFT of the following sequence

$$x(n) = \begin{cases} \dfrac{X(0)}{N} & \text{for } n = 0 \\ \dfrac{X(N-n)}{N} & \text{for } n = 1, 2, \ldots, N-1 \end{cases}$$

Thus, the first value is stored at address 0 while the rest are stored in reverse order. This operation can easily be implemented in hardware by changing the address lines to the memory or by using an up–down counter that is properly initiated.

Another method to compute the IFFT is to first interchange the real and imaginary parts, then perform the FFT, and, finally, interchange the real and imaginary parts. We will use this method in an implementation of an FFT processor.

FFT PROCESSOR—CASE STUDY 1

Develop an FFT processor usable VLSI building block.

In order to be flexible so that the processor, the performance in terms of computational throughput, word length, and transform length should be easily modifiable.

Specification

The processor shall compute a 1024-point complex FFTs with a continuous throughput of more than 2000 FFTs per second.

The processor shall also be able to perform both the FFT and the IFFT.

The host is a general purpose 32-bit computer.

We assume that the data rate is a modest 16 MHz.

The data word length for both input and output is selected to be 16 bits for the real and imaginary parts.

The internal data word length is, in this early stage of the design process, estimated to 21 bits.

The required coefficient word length is assumed to be14 bits

We arbitrarily select to implement the Sande–Tukey FFT.

System Design Phase

The first step in the system design phase is to partition the computation of an FFT into three consecutive processes;

- reading the input data from the host,
- performing the FFT/, and
- writing the result into the memory of the host.

The requirement is that the execution time for the FFT, including I/O, should not exceed 0.5 ms.

The I/O transfer frequency should be only 16 MHz. Input and output data to the FFT processor are transferred as two 16-bit real numbers.

A total of 1024 complex numbers are transferred to and from the FFT processor.

The time needed for the I/O is

$$t_{I/O} = \frac{2 \cdot 1024}{16 \cdot 10^6} = 0.128 \text{ ms}$$

It is possible to overlap the I/O operations, i.e., writing and reading data, and computation of the FFT, thereby effectively extending the available time for computing the FFT. For the sake of simplicity we assume that this possibility is not exploited.

The available time for the FFT is therefore 0.372 ms.

The I/O processes will handle both the rearranging of data that is required to compute the IFFT and the unscrambling of the data array in the last stage of the FFT.

The IFFT can be obtained by interchanging the real and imaginary parts, when the data are both written into and read from the memory.

The unscrambling can be accomplished by reversing the address lines to the memory when reading data out of the FFT processor.

Hence, both of these tasks can be accomplished without any extra processing time.

In fact, this is an example of merging operations and communications!