Institutionen för Reglerteknik
Tekniska Högskolan i Lund

Linear Systems I

Exam October, 1997

The time is 36 hours or less. Computers may be used and the books and notes may of course be consulted. You should ask me if anything is questionable or difficult to understand, but you may not help each other during the exam.

Participants:
Anna Delfin
Sven Hedlund
Fredrik Kahl
Michael Lantz
Jonas Svensson

Good Luck!
Per

1. Linearize the fermentation model

$$\frac{dV}{dt} = F$$
$$\frac{dVX}{dt} = \mu(G)VX$$
$$\frac{dVG}{dt} = -q_G(G)VX + G_{in}F$$
$$\mu = Y_x q_G$$
$$\mu = \mu_{max}\frac{G}{G+k_s}$$

with $Y_x = 0.5$ (g-cells per g-glucose), $\mu_{max} = 0.65h^{-1}$, $k_s = 0.01g/\ell$, $G_{in} = 500g/\ell$, $V(0) = 2\ell$, $VX(0) = 10g$, $G(0) = k_s$ around the nominal glucose feed $F^{\circ}(t) = F_0 e^{\mu_0 t}$, $\mu_0 = \mu(k_s)$, $F_0 G_{in} = q_G(k_s)VX(0)$.

Determine the reachability Gramian between $t_1 = 1$ and $t_2 = 2$. Discuss also how the "gain" and "timeconstant" of the glucose subsystem changes with time. (10 p)

2. Consider the periodic system

$$\dot{x}(t) = -(\sin t + 2)x(t)$$

with period $T = 2\pi$. Determine $\Phi(t,\tau)$ and a periodic Lyapunov transformation $x(t) = P(t)z(t)$ giving a timeinvariant z-system.

Would there exist initial conditions such that

$$\dot{x}(t) = -(\sin t + 2)x(t) + u(t)$$

has a periodic solution for $u(t) = \sin t$? (10 p)

3. An electrical system consists of three circuits, each with a resistor and an inductance in series. Assume also coupling between the inductances. Let the first circuit be connected to a voltage source, and let the other two circuits be closed. Thus the system can be described by

$$(sL + R)I(s) = e_1 U(s)$$

where L is a positive definite symmetric matrix of nonnegative inductances, R is a diagonal matrix of positive resistances, and $e_1^T = [1,0,0]$. Introduce a realization and formulate the PBH-test for controllability. Assume for simplicity, $L_{1,1} = 1$, $R_{1,1} = 1$. Discuss intuitive parameter combinations resulting in lack of controllability. There are actually also some nonintuitive combinations. Determine the reachable subspace and its dimension for

$$L = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 2 & 1/2 \\ 1/2 & 1/2 & 3 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \tag{1}$$

Try finally to get a general condition in terms of $L_{1,2}$, $L_{1,3}$, $L_{2,3}$, $L_{2,2}$, $L_{3,3}$ and $R_{2,2}$, $R_{3,3}$ (quite hard). (10 p)

4. Assume in the previous example with parameters (1) that $L_{3,3} = 3+1/1000$. Assume also zero initial currents. Consider the voltage function $\{u(t),\ t \in [0,\infty]\}$ required to achieve $i(\infty) = i_f$. Determine the function u_m with minimal 2-norm, i.e. minimizing $||u|| = \int_0^\infty u^2(t)dt$. Show how you may utilize the `lyap`-command in Matlab. Which combination of currents i_f requires the maximal and minimal $||u_m||$? (10 p)

5. A linearization of the quadruple-tank process is given by

$$\dot{x} = \begin{bmatrix} -1/T_1 & 0 & 1/T_3 & 0 \\ 0 & -1/T_2 & 0 & 1/T_4 \\ 0 & 0 & -1/T_3 & 0 \\ 0 & 0 & 0 & -1/T_4 \end{bmatrix} x + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & b_3 \\ b_4 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $T = [63, 91, 39, 56]$ and $b = [0.048, 0.035, 0.078, 0.056]$. Determine the controller form, and a state feedback making the poles twice as fast. Determine also a reduced order observer with poles at $s = -1/10$ and $s = -1/20$. Is the resulting controller, i.e. transfer function from y to u, reasonable? (10 p)

6. Use Rugh's method (Corollary 14.13) to get noninteracting control of the quadrupel tank. Determine the Markov parameters and the relative degrees. (5 p)

7. In the enclosed very recent paper is calculated among other things the singular values of the controllability matrix for a triple inverted pendulum. Assume that the system is initially at rest, except for a deviation of 5 degrees in the third link. Use the controller (18). Check the closed loop eigenvalues and determine the square integral of the control signal. Hint: Determine a Lyapunov equation for the integral

$$x_0^T \left(\int_0^\infty e^{(A+BK)^T t} K^T K e^{(A+BK)t} dt \right) x_0$$

Use Matlab, **c2d**, to sample the system with sampling interval $h = 12\text{ms}$. Minimize the control signal norm to reach the origin in 1000 sampling intervals, using the discrete time controllability Gramian $W_c(0, 1000)$. One idea would be to find a recursion for $W_c(0, k)$. (5 p)