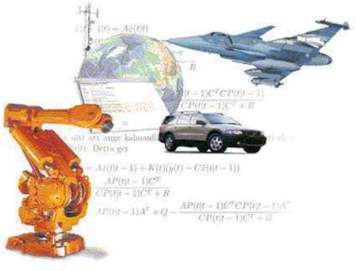
Iterative Learning Control – From a Controllability Point of View



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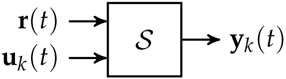
Outline

1. Introduction
2. State space model in the iteration domain
3. Controllability
4. Target path controllability
5. Lead-in
6. Conclusions

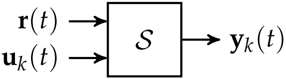
Introduction

- *Iterative learning control* (ILC) is a method to improve the control performance of iterative processes.
- System description at time t and ILC iteration k:



Introduction

- *Iterative learning control* (ILC) is a method to improve the control performance of iterative processes.
- System description at time t and ILC iteration k:



- ILC control signal update:

$$\mathbf{u}_{k+1}(t) = \mathcal{F}\left(\{\mathbf{u}_k(i)\}_{i=0}^{N-1}, \{\mathbf{e}_k(i)\}_{i=0}^{N-1}\right), \quad t = 0, \dots, N-1,$$

- Goal: find an update function $\mathcal{F}$ such that

$$\|\mathbf{e}_k(t)\| \to 0, \quad k \to \infty, \quad t = 0, \dots, N-1$$

- Convergence proofs usually use a batch formulation of the system

$$\overline{\mathbf{y}}_k = \mathbf{S_u}\overline{\mathbf{u}}_k + \mathbf{S_r}\bar{\mathbf{r}}.$$

- In literature, specific ILC algorithms are shown to give zero error if $\mathbf{S_u}$ has full row rank.

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- In literature, specific ILC algorithms are shown to give zero error if $\mathbf{S}_{\mathbf{u}}$ has full row rank.
- What does it mean that $\mathbf{S}_{\mathbf{u}}$ has full row rank?
- Controllability in the iteration domain will be used to answer the question.
- The result will be extended to *target path controllability* (TPC), and “lead-in” will be introduced.

State space model in the iteration domain

- Linear time-invariant state space model:

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B_u}\mathbf{u}(t) + \mathbf{B_r}\mathbf{r}(t),$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t).$$

- Batch formulation: $\overline{\mathbf{x}} = \begin{pmatrix}\mathbf{x}(1)^{\mathsf{T}} & \dots & \mathbf{x}(N)^{\mathsf{T}}\end{pmatrix}^{\mathsf{T}}$

$$\overline{\mathbf{x}} = \mathbf{\Phi}\mathbf{x}(0) + \mathbf{S_{xu}}\overline{\mathbf{u}} + \mathbf{S_{xr}}\overline{\mathbf{r}},$$
$$\overline{\mathbf{y}} = \boldsymbol{\mathcal{C}}\overline{\mathbf{x}}.$$

- At ILC iteration k and $k+1$ it holds

$$\overline{\mathbf{x}}_k = \mathbf{\Phi}\mathbf{x}(0) + \mathbf{S_{xu}}\overline{\mathbf{u}}_k + \mathbf{S_{xr}}\overline{\mathbf{r}},$$
$$\overline{\mathbf{x}}_{k+1} = \mathbf{\Phi}\mathbf{x}(0) + \mathbf{S_{xu}}\overline{\mathbf{u}}_{k+1} + \mathbf{S_{xr}}\overline{\mathbf{r}},$$

- hence

$$\overline{\mathbf{x}}_{k+1} = \overline{\mathbf{x}}_k + \mathbf{S_{xu}}(\overline{\mathbf{u}}_{k+1} - \overline{\mathbf{u}}_k) = \overline{\mathbf{x}}_k + \mathbf{S_{xu}}\Delta_{\overline{\mathbf{u}}_k}.$$

- Controllability matrix: $\mathscr{C} = \begin{pmatrix}\mathbf{S_{xu}} \cdots \mathbf{S_{xu}}\end{pmatrix}$

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Theorem (State Controllability)

The system in the iteration domain is controllable if and only if

$$\operatorname{rank} \mathscr{C} = \operatorname{rank} \mathbf{S_{xu}} = N n_x \quad \Leftrightarrow$$
$$\operatorname{rank} \mathbf{B_u} = n_x$$

Corollary (State Controllability)

A necessary condition for the system in the iteration domain to be controllable is that $n_u \geq n_x$.

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- Very strict requirement for controllability.
- For ILC it is more relevant to see if the output is controllable.

- Output controllability matrix: $\mathscr{C}^o = \boldsymbol{\mathcal{C}}\mathscr{C} = \left(\boldsymbol{\mathcal{C}}\mathbf{S}_{\mathbf{xu}} \cdots \boldsymbol{\mathcal{C}}\mathbf{S}_{\mathbf{xu}}\right)$

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Corollary (Output Controllability)

A necessary condition for the system in the iteration domain to be controllable is that $\operatorname{rank} \mathbf{B_u} \geq n_y$.

- Note that $\mathbf{S_u} = \boldsymbol{\mathcal{C}}\mathbf{S_{xu}}$, hence $\|\mathbf{e}_k(t)\| \to 0$ is possible if the system is output controllable.

Interpretation of controllability requirements

- A SISO system with state dimension n_x can require n_x time steps to reach the desired state.
  - $\Rightarrow$ the first part of $\bar{\mathbf{x}}$ cannot be defined arbitrary.

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- A SISO system with state dimension n_x can require n_x time steps to reach the desired state.
  - $\Rightarrow$ the first part of $\bar{\mathbf{x}}$ cannot be defined arbitrary.
- $\bar{\mathbf{x}}$ must follow the dynamics in the time domain.
  - Let $\mathbf{x} = \begin{pmatrix} p & v \end{pmatrix}^{\mathsf{T}}$ and

$$\mathbf{x}(t+1) = \begin{pmatrix} 1 & T_s \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} T_s^2/2 \\ T_s \end{pmatrix} u(t)$$

  - If $p(t) = a$ and $v(t) = b$ then

$$p(t+1) = a + bT_s + T_s^2/2u(t)$$
$$v(t+1) = b + T_s u(t)$$

  - Hence not possible to choose $\mathbf{x}(t+1)$ arbitrary with only $u(t)$ $\Rightarrow$ $\bar{\mathbf{x}}$ cannot be chosen arbitrary.

- Output controllability is achieved if p or v is chosen as output.
- If instead

$$\mathbf{x}(t) = \begin{pmatrix} p(t) \\ v(t) \end{pmatrix}$$

$$\mathbf{x}(t+1) = \begin{pmatrix} 1 & T_s \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ T_s \end{pmatrix} u(t)$$

then the requirement for output controllability is not satisfied if p is chosen as output.

- Extension to controllability to check if a system can follow a trajectory.
- The lead p states how many samples that might be required to go from the start point to the trajectory and the lag q states during how many samples the system can follow the trajectory.

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Theorem (Target Path Controllability)

An LTI system is TPC with lead p and lag q iff $\operatorname{rank} \mathbf{S_{yu}}(p,q) = qn_y$, where

$$\mathbf{S_{yu}}(p,q) = \begin{pmatrix} \mathbf{C}\mathbf{A}^{p-1}\mathbf{B_u} & \cdots & \mathbf{C}\mathbf{B_u} & \cdots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ \mathbf{C}\mathbf{A}^{p+q-2}\mathbf{B_u} & & \cdots & & \mathbf{C}\mathbf{B_u} \end{pmatrix}.$$

Remark

If $p = n_x$ and $q = 1$, then $\mathbf{S}_{\mathbf{yu}}(n_x, 1) = \mathscr{C}^o$, i.e., requirement for standard output controllability is obtained.

Theorem (Output Controllability vs. TPC)

Output controllability of the system in the iteration domain is equivalent to the system in the time domain being TPC with $p = 1$ and $q = N$.

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If $p = n_x$ and $q = 1$, then $\mathbf{S_{yu}}(n_x, 1) = \mathscr{C}^o$, i.e., requirement for standard output controllability is obtained.

Theorem (Output Controllability vs. TPC)

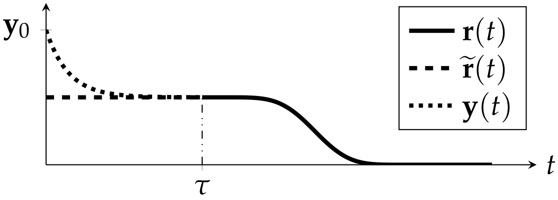
Output controllability of the system in the iteration domain is equivalent to the system in the time domain being TPC with $p = 1$ and $q = N$.

Given the model

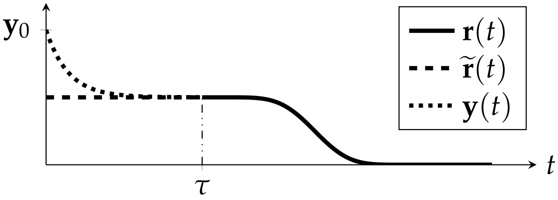
$$\mathbf{x}(t+1) = \begin{pmatrix} 1 & T_s \\ 0 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ T_s \end{pmatrix} u(t)$$

then $\mathbf{S_{yu}}(2, N-1)$ satisfies the requirement for TPC.

- Use TPC to find p.
- Move the reference path $\tau \geq p$ samples forward in time.



- Use TPC to find p.
- Move the reference path $\tau \geq p$ samples forward in time.
- Append a new initial reference trajectory $\widetilde{\mathbf{r}}(t)$.
- The output now follows the new reference and
  - $\mathbf{e}(t)$ for $t \leq \tau$ is not of importance.
  - $\mathbf{e}(t) \to 0$ for $t > \tau$, given an ILC algorithm that makes the error converge to 0.



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Conclusions

- Assumptions of the system used for convergence proofs are often too strict to be useful in practice.
- A commonly used requirement is equivalent to output controllability for a state space model in the iteration domain.
- Target path controllability of the system in the time domain is more relevant to investigate than controllability of the system in the iteration domain.

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- Assumptions of the system used for convergence proofs are often too strict to be useful in practice.
- A commonly used requirement is equivalent to output controllability for a state space model in the iteration domain.
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Fundamental properties of a system

$\Rightarrow$

limitations of trajectories that can be tracked using ILC.

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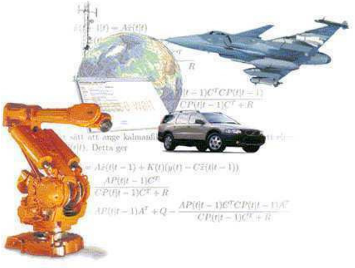

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