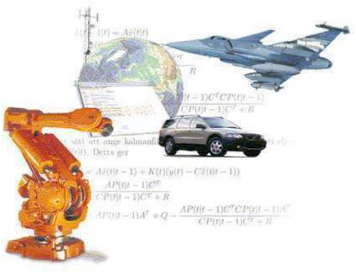
ML Estimation of Process Noise Variance in Dynamic Systems



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Outline

1. Problem Formulation
2. Estimation of process noise covariance
3. Alternative Methods
4. Simulation Results

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Background

- The elasticity (in joints, arms, . . .) for an industrial robot has increased over the last years.
- The motion control must be improved for these new robots.

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- The motion control must be improved for these new robots.
- One solution can be to estimate the tool position and use that estimate in the feedback loop.
- The tool position can be estimated with e.g. a EKF which uses knowledge about the process noise.

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- The elasticity (in joints, arms, ...) for an industrial robot has increased over the last years.
- The motion control must be improved for these new robots.
- One solution can be to estimate the tool position and use that estimate in the feedback loop.
- The tool position can be estimated with e.g. a EKF which uses knowledge about the process noise.
- The accuracy is very important, < 1 mm.

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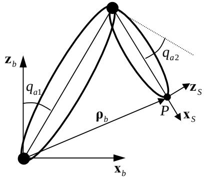
- Serial robot with two degrees of freedom.
- Nonlinear stiffness.
- Nonlinear friction.



$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) + T(q) + D\dot{q} + F(\dot{q}) = u$$

- The model structure is common for mechanical systems derived by Newton's second law of motion or Lagrange's equation.

Problem Formulation

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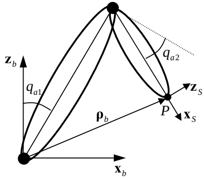

$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) + T(q) + D\dot{q} + F(\dot{q}) = u$$

- The model structure is common for mechanical systems derived by Newton's second law of motion or Lagrange's equation.
- Accelerometer

$$\ddot{\rho}_s(q_a,\dot{q}_a,\ddot{q}_a) = R_s^b(q_a)\Big(\ddot{\rho}_b(q_a,\dot{q}_a,\ddot{q}_a) + G_b\Big) + \delta_s$$

- States

$$x = \begin{pmatrix} q_a^T & q_m^T & \dot{q}_a^T & \dot{q}_m^T \end{pmatrix}^T \in \mathbb{R}^8$$

- Discrete state space model (cont. $\rightarrow$ disc. using Euler forward)

$$x_{k+1} = F_1(x_k, u_k) + F_2(x_k)v_k, \quad v_k \sim \mathcal{N}(0, Q)$$
$$y_k = \begin{pmatrix} q_{m,k} \\ \ddot{\rho}_{s,k} \end{pmatrix} + e_k = h(x_k, u_k) + e_k, \quad e_k \sim \mathcal{N}(0, R)$$

where

$$F_2(x_k) = \begin{pmatrix} 0 \\ \widetilde{F}_2(x_k) \end{pmatrix}.$$

- All model parameters known except for the covariance matrix Q.

- Maximum Likelihood (ML) method

$$\hat{\theta}^{ML} = \arg\max_{\theta \in \Theta} \log p_\theta(y_{1:N}).$$

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- Solution hard to find $\Rightarrow$ Expectation Maximisation (EM) algorithm.
  1. Select an initial value θ_0 and set $l = 0$.
  2. Expectation Step (E-step): Calculate

$$\Gamma(\theta; \theta_l) = E_{\theta_l}\left[\log p_\theta(y_{1:N}, x_{1:N}) | y_{1:N}\right].$$

  3. Maximisation Step (M-step): Compute

$$\theta_{l+1} = \arg\max_{\theta \in \Theta} \Gamma(\theta; \theta_l).$$

  4. If converged, stop. If not, set $l = l + 1$ and go to step 2.

Outline

- Estimate Q with the Expectation Maximisation (EM) algorithm.

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- Main ideas
  - Calculate $\Gamma(Q; Q_l)$ using linearisation.
  - Use the smoothed states (EKS) as approximation when necessary. Possible since the smoothed densities are peaky when the SNR is high which is the case here.

- Conditional densities

$$x_{k+1} \sim p(x_{k+1}|x_k) = \mathcal{N}\left(x_{k+1}; F_{1,k}, F_{2,k} Q F_{2,k}^T\right)$$
$$y_k \sim p(y_k|x_k) = \mathcal{N}\left(y_k; h_k, R\right)$$

- Conditional densities

$$
\begin{aligned}
x_{k+1} &\sim p(x_{k+1}|x_k) = \mathcal{N}\left(x_{k+1}; F_{1,k}, F_{2,k} Q F_{2,k}^T\right) \\
y_k &\sim p(y_k|x_k) = \mathcal{N}\left(y_k; h_k, R\right)
\end{aligned}
$$

- The joint log likelihood function

$$
\begin{aligned}
L_Q(y_{1:N}, x_{1:N}) = \log p_Q(y_{1:N}, x_{1:N}) = \widetilde{L} \\
- \frac{1}{2}\sum_{i=2}^{N} (x_i - F_{1,i-1})^T \left(F_{2,i-1} Q F_{2,i-1}^T\right)^{\dagger} (x_i - F_{1,i-1}) \\
+ \frac{1}{2}\sum_{i=2}^{N} \log\left(\prod_{\lambda_j \neq 0} \lambda_j\left(\left(F_{2,i-1} Q F_{2,i-1}^T\right)^{\dagger}\right)\right)
\end{aligned}
$$

Estimation of process noise covariance

- Conditional densities

$$x_{k+1} \sim p(x_{k+1}|x_k) = \mathcal{N}\left(x_{k+1}; F_{1,k}, F_{2,k} Q F_{2,k}^T\right)$$
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- The joint log likelihood function

$$\begin{aligned} L_Q(y_{1:N}, x_{1:N}) &= \log p_Q(y_{1:N}, x_{1:N}) = \widetilde{L} \\ &- \frac{1}{2} \sum_{i=2}^{N} (x_i - F_{1,i-1})^T \left(F_{2,i-1} Q F_{2,i-1}^T\right)^{\dagger} (x_i - F_{1,i-1}) \\ &+ \frac{1}{2} \sum_{i=2}^{N} \log\left(\left|\left(\widetilde{F}_{2,i-1} Q \widetilde{F}_{2,i-1}^T\right)^{\dagger}\right|\right) \end{aligned}$$

- Expectation of the joint log likelihood function

$$
\begin{aligned}
\Gamma(Q;Q_l) &= E_{Q_l}\left[L_Q(y_{1:N}, x_{1:N}) | y_{1:N}\right] = \bar{L} \\
&+ \frac{1}{2}\sum_{i=2}^{N} E_{Q_l}\left[\log\left(\left|\left(\widetilde{F}_{2,i-1} Q \widetilde{F}_{2,i-1}^T\right)^{\dagger}\right|\right) \middle| y_{1:N}\right] \\
&- \frac{1}{2}\operatorname{tr}\sum_{i=2}^{N} E_{Q_l}\left[\left(F_{2,i-1} Q F_{2,i-1}^T\right)^{\dagger} \left(x_i - F_{1,i-1}\right)\left(x_i - F_{1,i-1}\right)^T \middle| y_{1:N}\right]
\end{aligned}
$$

- Calculation of the first expectation

$$E_{Q_l}\left[\log\left(\left|\left(\widetilde{F}_{2,i-1}Q\widetilde{F}_{2,i-1}^T\right)^\dagger\right|\right)\middle| y_{1:N}\right]$$
$$= \int \log\left(\left|\left(\widetilde{F}_2(x_{i-1})Q\widetilde{F}_2^T(x_{i-1})\right)^\dagger\right|\right)p_{Q_l}(x_{i-1}|y_{1:N})\,\mathrm{d}x_{i-1}$$

Estimation of process noise covariance
cont'd

- Calculation of the first expectation

$$E_{Q_l}\left[\log\left(\left|\left(\widetilde{F}_{2,i-1}Q\widetilde{F}^T_{2,i-1}\right)^{\dagger}\right|\right)\middle| y_{1:N}\right]$$
$$= \int \log\left(\left|\left(\widetilde{F}_2(x_{i-1})Q\widetilde{F}^T_2(x_{i-1})\right)^{\dagger}\right|\right)p_{Q_l}(x_{i-1}|y_{1:N})\,\mathrm{d}x_{i-1}$$

- Cannot be solved $\Rightarrow$ Approximation: Use the smoothed states.

$$E_{Q_l}\left[\log\left(\left|\left(\widetilde{F}_{2,i-1}Q\widetilde{F}^T_{2,i-1}\right)^{\dagger}\right|\right)\middle| y_{1:N}\right]$$
$$\approx \log\left(\left|\left(\widetilde{F}_2(\hat{x}^s_{i-1|N})Q\widetilde{F}^T_2(\hat{x}^s_{i-1|N})\right)^{\dagger}\right|\right)$$

- Calculation of the second expectation

$$E_{Q_l}\left[\left(F_{2,i-1}QF_{2,i-1}^T\right)^{\dagger}(x_i - F_{1,i-1})(x_i - F_{1,i-1})^T \middle| y_{1:N}\right]$$

$$= \int \left(F_2(x_{i-1})QF_2^T(x_{i-1})\right)^{\dagger}(x_i - F_{1,i-1})(x_i - F_{1,i-1})^T$$

$$\times\, p_{Q_l}(x_i, x_{i-1}|y_{1:N})\,\mathrm{d}x_i\,\mathrm{d}x_{i-1}$$

- Calculation of the second expectation

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$$\times\, p_{Q_l}(x_i, x_{i-1}|y_{1:N})\, \mathrm{d}x_i\, \mathrm{d}x_{i-1}$$

- Use the smoothed states again

$$E_{Q_l}\left[\left(F_{2,i-1}QF_{2,i-1}^T\right)^{\dagger}(x_i - F_{1,i-1})(x_i - F_{1,i-1})^T \middle| y_{1:N}\right]$$
$$\approx \left(F_2(\hat{x}^s_{i-1|N})QF_2^T(\hat{x}^s_{i-1|N})\right)^{\dagger}$$
$$\times \int (x_i - F_{1,i-1})(x_i - F_{1,i-1})^T p_{Q_l}(x_i, x_{i-1}|y_{1:N})\, \mathrm{d}x_i\, \mathrm{d}x_{i-1}$$

- A first order Taylor approximation gives

$$
\begin{aligned}
M &\triangleq \int (x_i - F_{1,i-1}) (x_i - F_{1,i-1})^T p_{Q_l}(x_i, x_{i-1} | y_{1:N}) \, \mathrm{d}x_i \, \mathrm{d}x_{i-1} \\
&= \begin{pmatrix} -J_1 & I \end{pmatrix} P^{\xi,s}_{i|N} \begin{pmatrix} -J_1 & I \end{pmatrix}^T \\
&\quad + \left(\hat{x}^s_{i|N} - F_1(\hat{x}^s_{i-1|N})\right) \left(\hat{x}^s_{i|N} - F_1(\hat{x}^s_{i-1|N})\right)^T \\
J_1 &= \left. \frac{\partial F_1(x)}{\partial x} \right|_{x=\hat{x}^s_{i-1|N}}
\end{aligned}
$$

- $P^{\xi,s}_{i|N}$ is the smoothed state covariance for the augmented state vector $\xi = \begin{pmatrix} x^T_{i-1} & x^T_i \end{pmatrix}^T$

- Finally

$$\Gamma(Q;Q_l) = \bar{\bar{\bar{L}}} + \frac{N-1}{2}\log\left|Q^{-1}\right| - \frac{1}{2}\operatorname{tr} Q^{-1}W$$

where

$$W = \sum_{i=2}^{N} F_2^{\dagger}(\hat{x}^s_{i-1|N}) M \left(F_2^{\dagger}(\hat{x}^s_{i-1|N})\right)^T$$

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- Maximisation of $\Gamma(Q;Q_l)$ w.r.t. Q gives

$$Q_{l+1} = \frac{1}{N-1}\sum_{i=2}^{N} F_2^{\dagger}(\hat{x}^s_{i-1|N}) M \left(F_2^{\dagger}(\hat{x}^s_{i-1|N})\right)^T$$

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Alt. 1 Minimisation of the estimation error, given by e.g. EKF, w.r.t. Q, where Q is parametrised as a diagonal matrix.

Alt. 2 Calculate v_k from the state space model.

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Alt. 2 Calculate v_k from the state space model.

1. Select an initial value Q_0 and set $l = 0$.
2. Use the EKS with Q_l.
3. Calculate the noise according to

$$v_k = F_2^{\dagger}\left(\hat{x}^s_{k|N}\right)\left(\hat{x}^s_{k+1|N} - F_1\left(\hat{x}^s_{k|N}, u_k\right)\right).$$

4. Let Q_{l+1} be the covariance matrix for v_k according to

$$Q_{l+1} = \frac{1}{N}\sum_{k=1}^{N} v_k^T v_k.$$

5. If converged, stop, If not, set $l = l + 1$ and go to step 2.

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- Monte Carlo simulations on one simulated path with different initial values.
- No true values for $Q \Rightarrow$ Use Q in an EKF and calculate the estimation error (RMSE) for the path.

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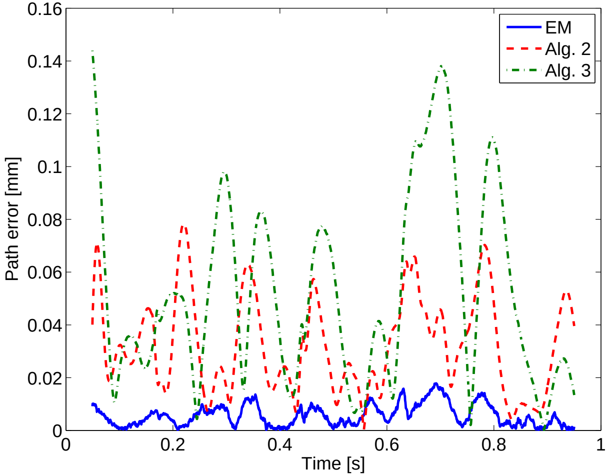
- No true values for $Q \Rightarrow$ Use Q in an EKF and calculate the estimation error (RMSE) for the path.
- The EM algorithm converges to the same RMSE for all initial values. Same thing happens for Alt. 2. Alt. 1 ends up in different errors.
- The EM algorithm gives the lowest RMSE (2-norm of the RMSE).

	Max	Min
EM	0.2999	0.2996
Alt. 1	3.3769	1.5867
Alt. 2	2.6814	2.6814



Path error [mm]
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
Time [s]
0
0.2
0.4
0.6
0.8
1
EM
Alg. 2
Alg. 3

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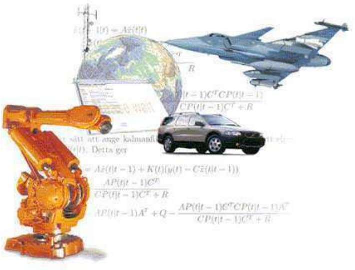

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