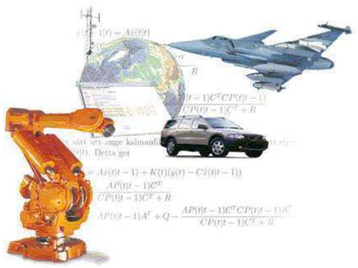
Robust Multivariable Control

Lecture 5



Anders Helmersson
anders.helmersson@liu.se

ISY/Reglerteknik
Linköpings universitet

Today's topics

- State-space feedback H_∞
- A special case
- A more general case

H_∞ control – simplified assumptions

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

Find a controller K that minimizes the H_∞ norm from w to z.

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + \qquad D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{1}$$

We start by assuming that $D_{12}^T D_{12} = I$ and $D_{12}^T C_1 = 0$.

The norm of z:

$$z^T z = \left(x^T C_1^T + u^T D_{12}^T\right)(C_1 x + D_{12} u) = x^T C_1^T C_1 x + u^T u.$$

LQR – Linear Quadratic Regulator

Introduce $V(x) = x^T P x$:

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax + B_1 w + B_2 u)^{\mathrm{T}} P x + x^T P (Ax + B_1 w + B_2 u)\end{aligned}$$

Compare with the LQR problem:

$$\int \dot{V}(x) + \underbrace{y^T y + u^T u}_{\text{criterion function}} \; dt = V(x(\infty)) - V(x(0)) + \|y\|^2 + \|u\|^2 = 0.$$

Here we have $\dot{V}(x) + z^T z - \gamma^2 w^T w$ since we want to minimize $\|z\|_2^2 - \gamma^2 \|w\|_2^2 \leq 0$ with respect to γ ($= \|.\|_\infty$).

$$\dot{V}(x) + z^T z - \gamma^2 w^T w = \dot{x}^T P x + x^T P \dot{x} + x^T C_1^T C_1 x + u^T u - \gamma^2 w^T w$$

$$= x^T \left(A^T P + PA + C_1^T C_1 \right) x + u^T u - \gamma^2 w^T w$$

$$+ w^T B_1^T P x + u^T B_2^T P x + x^T P B_1 w + x^T P B_2 u$$

[completing the squares]

$$= x^T \left(A^T P + PA + C_1^T C_1 \right) x + (u + B_2^T P x)^T (u + B_2^T P x) - x^T P B_2 B_2^T P x$$

$$- \gamma^2 (w - \gamma^{-2} B_1^T P x)^T (w - \gamma^{-2} B_1^T P x) + \gamma^{-2} x^T P B_1 B_1^T P x$$

$$= x^T \underbrace{\left(A^T P + PA + C_1^T C_1 - P B_2 B_2^T P + \gamma^{-2} P B_1 B_1^T P \right)}_{\text{Riccati equation = 0}} x$$

$$+ (u + B_2^T P x)^T \underbrace{(u + B_2^T P x)}_{v} - \gamma^2 (w - \gamma^{-2} B_1^T P x)^T \underbrace{(w - \gamma^{-2} B_1^T P x)}_{r}$$

Introduce $v = u + B_2 P x$ and $r = w - \gamma^{-2} B_1 P x$. This yields

$$J = \underbrace{\int_0^\infty \dot{V}(x)dt}_{V(x(\infty))-V(x(0))=0} + \underbrace{\|z\|_2^2 - \gamma^2\|w\|_2^2}_{\leq 0 \text{ if } H_\infty \text{ gain} \leq \gamma} = \|v\|_2^2 - \gamma^2\|r\|_2^2$$

Interpretation:

$u = -B_2^T P x$ ($v = 0$) is the best control signal that minimizes J.

$w = \gamma^{-2} B_1^T P x$ ($r = 0$) is the worst disturbance that maximizes J.

Use $u = -B_2^T P x$ as a state space feedback.

The Riccati equation

$$A^T X + XA + C_1^T C_1 - XB_2 B_2^T X + \gamma^{-2} XB_1 B_1^T X = 0$$

This gives

$$H_\infty(\gamma) = \begin{bmatrix} A & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix} = \begin{bmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{bmatrix}$$

We solved this equation during lecture 4. Find the stable eigenvalues of H_∞

$$H_\infty \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Lambda$$

and form $X = x_2 x_1^{-1}$, which assumes that H_∞ has no eigenvalues on the imaginary axis and that x_1 is invertible. Then $A + RX$ stable.

Note that if $\gamma \to \infty$ then $\gamma^{-2} XB_1 B_1^T X$ disappears and H_2 problem (LQR) is recovered.

Compare with the H_∞-norm (from lecture 2)

If H_∞ as defined below has no eigenvalues on the imaginary axis then $\|G\|_\infty < \gamma$.

$$H_\infty(\gamma) = \begin{bmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^TC - C^TDR^{-1}D^TC & -A^T - C^TDR^{-1}B^T \end{bmatrix}$$

with $R = \gamma^2 I - D^T D \succ 0$ ($\gamma > \|D\|$).

If $D = 0$ then

$$H_\infty(\gamma) = \begin{bmatrix} A & \gamma^{-2}BB^T \\ -C^TC & -A^T \end{bmatrix}$$

Conditions for the existence of a solution

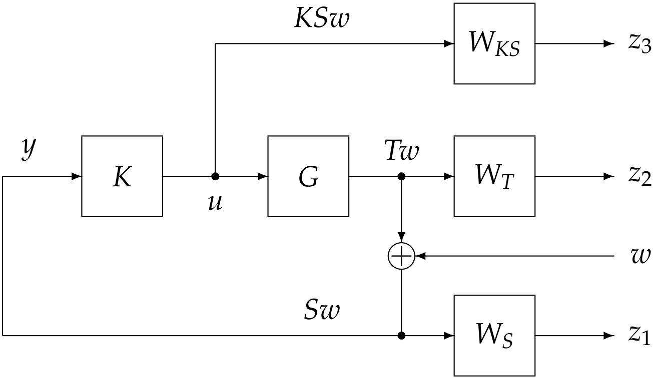
The Riccati equation has a solution if $H_\infty(\gamma) \in \mathrm{dom}(\mathrm{Ric})$. This means that

- H_∞ has no imaginary eigenvalues
- x_1 is invertible (non-singular)

In addition we require stability, $X \succeq 0$ (positive semi-definite).

We are looking for the smallest γ (or at least a γ that is small enough) that satisfies $H_\infty(\gamma) \in \mathrm{dom}(\mathrm{Ric})$. For instance, we can use bisection to find it (faster methods exist).

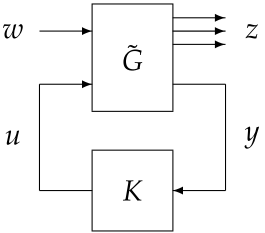
A special case



A special case

$$\tilde{G} = \begin{bmatrix} W_S & W_S G \\ 0 & W_T G \\ 0 & W_{KS} \\ I & G \end{bmatrix}$$

This system can be build using `sysic` or simulink. Avoid redundant states!



The innovation form

We can write $\tilde{G}$ on innovation form (Disturbance Feedforward, DF) under certain conditions:

$$\tilde{G} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & * \\ C_2 & I & 0 \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & I & 0 \end{array}\right]$$

Observer

$$\tilde{G}: \begin{cases} \dot{x} = Ax & +B_1 w & +B_2 u \\ z = C_1 x & & +D_{12} u \\ y = C_2 x & +w & \end{cases}$$

We can design a regulator using the observer

$$\dot{\hat{x}} = A\hat{x} + B_1 w + B_2 u,$$

which we use for generating the control signal $u = -B_2^T X \hat{x}$:

$$\begin{cases} \dot{\hat{x}} = (A - B_2 B_2^T X)\hat{x} & +B_1 w \\ u = -B_2^T X \hat{x} & \end{cases}$$

with $w = y - C_2 x$ (we replace $C_2 x$ with $C_2 \hat{x}$ in the observer)

$$\dot{\hat{x}} = (A - B_2 B_2^T X)\hat{x} + B_1(y - C_2\hat{x}) = (A - B_2 B_2^T X - B_1 C_2)\hat{x} + B_1 y$$

Observer error

$$
\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u \\
\dot{\hat{x}} &= (A - B_2 B_2^T X - B_1 C_2)\hat{x} + B_1 y \\
u &= -B_2^T X \hat{x} \\
w &= y - C_2 x.
\end{aligned}
$$

The model error, $\tilde{x} = \hat{x} - x$, is described by

$$\dot{\tilde{x}} = (A - B_1 C_2)\tilde{x}$$

We must make certain that $A - B_1 C_2$ is stable!

$$K = \left[\begin{array}{c|c} A - B_1 C_2 - B_2 B_2^T X & B_1 \\ \hline -B_2^T X & 0 \end{array}\right].$$

The DF case – summary

If the system is on DF form (Disturbance Feedforward)

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & I & 0 \end{array}\right] \quad \text{and} \quad D_{12}^{\mathrm{T}} \left[\begin{array}{cc} D_{12} & C_1 \end{array}\right] = \left[\begin{array}{cc} I & 0 \end{array}\right].$$

Let

$$H_\infty = \left[\begin{array}{cc} A & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{array}\right] \in \mathrm{dom}(\mathrm{Ric})$$

and $X = \mathrm{Ric}\, H_\infty \succeq 0$. If $A - B_1 C_2$ is stable, then the controller is given by

$$K = \left[\begin{array}{c|c} A - B_1 C_2 - B_2 B_2^T X & B_1 \\ \hline -B_2^T X & 0 \end{array}\right].$$

A more general case

Original system

$$G : \begin{cases} \dot{x} = Ax & +B_1 w & +B_2 u \\ z = C_1 x & & +D_{12} u \\ y = C_2 x & +D_{21} w & \end{cases}$$

Introduce r and v as two artificial signals in the "shadow systems":

$$P : \begin{cases} \dot{x} = Ax & +B_1 w & +B_2 u \\ z = C_1 x & & +D_{12} u \\ r = -\gamma^{-2} B_1^T X x & +w & \end{cases}$$

$$G_{\mathrm{tmp}} : \begin{cases} \dot{x} = Ax & +B_1 w & +B_2 u \\ v = B_2^T X x & & +u \\ y = C_2 x & +D_{21} w & \end{cases}$$

Replace $u = v - B_2^T X x$:

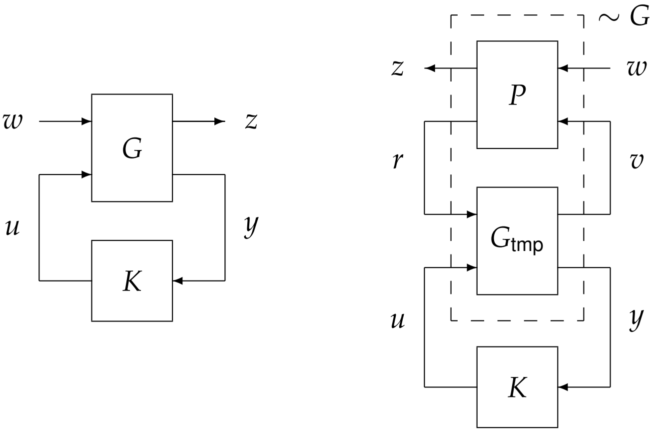
$$P : \begin{cases} \dot{x} = (A - B_2 B_2^T X)x & +B_1 w & +B_2 v \\ z = (C_1 - D_{12} B_2^T X)x & & +D_{12} v \\ r = -\gamma^{-2} B_1^T X x & +w & \end{cases}$$

Replace $w = r + \gamma^{-2} B_1^T X x$:

$$G_{\text{tmp}} : \begin{cases} \dot{x} = (A + \gamma^{-2} B_1 B_1^T X)x & +B_1 r & +B_2 u \\ v = B_2^T X x & & +u \\ y = (C_2 + \gamma^{-2} \underbrace{D_{21} B_1^T}_{=0} X)x & +D_{21} r & \end{cases}$$

Here we assume that $D_{21} D_{21}^T = I$ and $D_{21} B_1^T = 0$.

Rewriting G



Dual problem: DF $\leftrightarrow$ OE

$$P = \left[\begin{array}{c|cc} A - B_2 B_2^T X & B_1 & B_2 \\ \hline C_1 - D_{12} B_2^T X & 0 & D_{12} \\ -\gamma^{-2} B_1^T X & I & 0 \end{array}\right]$$

$$G_{\mathsf{tmp}} = \left[\begin{array}{c|cc} A + \gamma^{-2} B_1 B_1^T X & B_1 & B_2 \\ \hline B_2^T X & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

Here G_{tmp} is on a form that is called “Output Estimation”, OE, which is dual to DF:

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & I & 0 \end{array}\right]$$

We have already solved this special case!

Deriving the controller

The dual of G_{tmp}:

$$G_{\text{tmp,DUAL}} = \left[\begin{array}{c|cc} A^T + \gamma^{-2} X B_1 B_1^T & X B_2 & C_2^T \\ \hline B_1^T & 0 & D_{21}^T \\ B_2^T & I & 0 \end{array}\right]$$

The controller of the dual

$$K_{\text{DUAL}} = \left[\begin{array}{c|c} A^T + \gamma^{-2} X B_1 B_1^T - X B_2 B_2 - C_2^T C_2 Y_{\text{tmp}} & X B_2 \\ \hline -C_2 Y_{\text{tmp}} & 0 \end{array}\right].$$

Back to normal form

$$K = \left[\begin{array}{c|c} A + \gamma^{-2} B_1 B_1^T X - B_2 B_2^T X - Y_{\text{tmp}} C_2^T C_2 & -Y_{\text{tmp}} C_2^T \\ \hline B_2^T X & 0 \end{array}\right].$$

It remains to check $A - B_1 C_2$

For DF there was a condition that $A - B_1 C_2$ must be stable. For OE case this corresponds to that $A^T - C_1^T B_2^T$, or $A - B_2 C_1$, must be stable.

Apply this on G_{tmp} ($A_{\text{tmp}} = A + \gamma^{-2} B_1 B_1^{\mathrm{T}} X$, $B_{\text{tmp},2} = B_2$ and $C_{\text{tmp},1} = B_2^{\mathrm{T}} X$):

$$A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X$$

shall be stable.

This is identical to that $A + RX$ is stable for H_∞ (see page 10, state-space feedback).

$$G_{\mathsf{tmp}} = \left[\begin{array}{c|cc} A + \gamma^{-2} B_1 B_1^T X & B_1 & B_2 \\ \hline B_2^T X & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

The Hamiltonian for this problem becomes

$$J_{\mathsf{tmp}} = \begin{bmatrix} (A + \gamma^{-2} B_1 B_1^T X)^T & \gamma^{-2} X B_2 B_2^T X - C_2^T C_2 \\ -B_1 B_1^T & -(A + \gamma^{-2} B_1 B_1^T X) \end{bmatrix}$$

Compare to previous results

$$H_\infty = \begin{bmatrix} A & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix}$$

Not yet symmetric! $(A, B, C) \leftrightarrow (A^{\mathrm{T}}, C^{\mathrm{T}}, B^{\mathrm{T}})$

Similarity transformation on J_{tmp} to obtain J_∞

$$J_{\text{tmp}} = \begin{bmatrix} (A+\gamma^{-2}B_1B_1^TX)^T & \gamma^{-2}XB_2B_2^TX - C_2^TC_2 \\ -B_1B_1^T & -(A+\gamma^{-2}B_1B_1^TX) \end{bmatrix}$$

$$J_\infty = T^{-1}J_{\text{tmp}}T = \begin{bmatrix} I & \gamma^{-2}X \\ 0 & I \end{bmatrix} J_{\text{tmp}} \begin{bmatrix} I & -\gamma^{-2}X \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A^T & \gamma^{-2}XB_2B_2^TX - C_2^TC_2 - \gamma^{-2}X(A+\gamma^{-2}B_1B_1^TX) \\ -B_1B_1^T & -(A+\gamma^{-2}B_1B_1^TX) \end{bmatrix} \begin{bmatrix} I & -\gamma^{-2}X \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A^T & -C_2^TC_2 - \gamma^{-2}(A^TX + XA + \gamma^{-2}XB_1B_1^TX - XB_2B_2^TX) \\ -B_1B_1^T & -A \end{bmatrix}$$

[use $A^TX + XA + C_1^TC_1 - XB_2B_2^TX + \gamma^{-2}XB_1B_1^TX = 0$ from H_∞]

$$= \begin{bmatrix} A^T & \gamma^{-2}C_1^TC_1 - C_2^TC_2 \\ -B_1B_1^T & -A \end{bmatrix}$$

Condition on XY

$$\mathcal{X}_-(J_{\mathsf{tmp}}) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \sim \begin{bmatrix} I \\ Y_{\mathsf{tmp}} \end{bmatrix}$$

$$\begin{aligned} \mathcal{X}_-(J_{\mathsf{tmp}}) &= T\mathcal{X}_-(J_\infty) = T \begin{bmatrix} I \\ Y \end{bmatrix} \\ &= \begin{bmatrix} I & -\gamma^{-2}X \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ Y \end{bmatrix} = \begin{bmatrix} I - \gamma^{-2}XY \\ Y \end{bmatrix} \end{aligned}$$

Thus, $I - \gamma^{-2}XY$ must be invertible, or, $\rho(XY) < \gamma^2$. Note that the eigenvalues of XY are real and non-negative.

$$Y_{\mathsf{tmp}} = Y(I - \gamma^2 XY)^{-1} = (I - \gamma^2 YX)^{-1} Y$$

Summary

Assume that

$$D_{11} = 0$$
$$D_{12}^T \begin{bmatrix} D_{12} & C_1 \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}$$
$$D_{21} \begin{bmatrix} D_{21}^T & B_1^T \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}$$

Conditions for the existence of an H_∞ controller with a closed-loop gain less than γ:

- $H_\infty \in \mathrm{dom(Ric)}$
- $X_\infty = \mathrm{Ric}\, H_\infty \succeq 0$
- $J_\infty \in \mathrm{dom(Ric)}$
- $Y_\infty = \mathrm{Ric}\, J_\infty \succeq 0$
- $\rho(X_\infty Y_\infty) < \gamma^2$

The controller

$$K = \left[\begin{array}{c|c} A + \gamma^{-2} B_1 B_1^{\mathrm{T}} X_\infty - B_2 B_2^{\mathrm{T}} X_\infty - (I - \gamma^2 Y_\infty X_\infty)^{-1} Y_\infty C_2^{\mathrm{T}} C_2 & (I - \gamma^2 Y_\infty X_\infty)^{-1} Y_\infty C_2^{\mathrm{T}} \\ \hline B_2^{\mathrm{T}} X_\infty & 0 \end{array}\right]$$

The general case

In the general case we can remove some of the conditions that we have assumed so far. For instance, that D_{11} and D_{22} must be zero, see 17.1-17.3 in ZDG.

We can remove the condition on $D_{22} = 0$ by first designing a controller with $D_{22} = 0$ and then use $K(I + D_{22}K)^{-1}$ as controller. This assumes that $I + D_{22}K(\infty)$ is invertible.

The general case

Assume that D_{12} is full column rank

$$H_\infty = \begin{bmatrix} A & 0 \\ 0 & -A^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & -C_1^{\mathrm{T}} \end{bmatrix} \left[\begin{array}{cc|c} -\gamma^2 I & 0 & D_{11}^{\mathrm{T}} \\ 0 & 0 & D_{12}^{\mathrm{T}} \\ \hline D_{11} & D_{12} & -I \end{array}\right]^{-1} \begin{bmatrix} 0 & B^{\mathrm{T}} \\ C_1 & 0 \end{bmatrix}$$

and assume that D_{21} is full row rank

$$J_\infty = \begin{bmatrix} A^{\mathrm{T}} & 0 \\ 0 & -A \end{bmatrix} - \begin{bmatrix} C^{\mathrm{T}} & 0 \\ 0 & -B_1 \end{bmatrix} \left[\begin{array}{cc|c} -\gamma^2 I & 0 & D_{11} \\ 0 & 0 & D_{21} \\ \hline D_{11}^{\mathrm{T}} & D_{21}^{\mathrm{T}} & -I \end{array}\right]^{-1} \begin{bmatrix} 0 & C \\ B_1^{\mathrm{T}} & 0 \end{bmatrix}$$

If D_{12} or D_{21} loses rank more analysis is needed (generalized eigenvalue problems). The problem can be solvable or it can be badly formulated (the controller gain goes to infinity as γ approaches its optimal value).

The general case

Generalized eigenvalue problems: (controller)

$$\left[\begin{array}{cc|ccc} & & A-\lambda I & B_1 & B_2 \\ & -\gamma & C_1 & D_{11} & D_{12} \\ \hline A^{\mathrm{T}}+\lambda I & C_1 & & & \\ B_1^{\mathrm{T}} & D_{11} & & -\gamma & \\ B_2^{\mathrm{T}} & D_{12} & & & \end{array}\right]\xi = 0$$

and (observer)

$$\left[\begin{array}{ccc|cc} & & & A-\lambda I & B_1 \\ & -\gamma & & C_1 & D_{11} \\ & & & C_2 & D_{21} \\ \hline A^{\mathrm{T}}+\lambda I & C_1^{\mathrm{T}} & C_2^{\mathrm{T}} & & \\ B_1^{\mathrm{T}} & D_{11}^{\mathrm{T}} & D_{21}^{\mathrm{T}} & & -\gamma \end{array}\right]v = 0$$

The general case

Compare these with the closed loop:

$$
\left[\begin{array}{ccccc|ccccc}
 & & & & & A-\lambda I & B_1 & B_2 & & \\
 & -\gamma & & & & C_1 & D_{11} & D_{12} & & \\
 & & & & & C_2 & D_{21} & D_{22} & & -I \\
 & & & & & & & & \hat{A}-\lambda I & \hat{B} \\
 & & & & & & & -I & \hat{C} & \hat{D} \\
\hline
A^{\mathrm{T}}+\lambda I & C_1^{\mathrm{T}} & C_2^{\mathrm{T}} & & & & & & & \\
B_1^{\mathrm{T}} & D_{11}^{\mathrm{T}} & D_{21}^{\mathrm{T}} & & & & -\gamma & & & \\
B_2^{\mathrm{T}} & D_{12}^{\mathrm{T}} & D_{22}^{\mathrm{T}} & & -I & & & & & \\
 & & & \hat{A}^{\mathrm{T}}+\lambda I & \hat{C}^{\mathrm{T}} & & & & & \\
 & & -I & \hat{B}^{\mathrm{T}} & \hat{D}^{\mathrm{T}} & & & & &
\end{array}\right]
$$

Anders Helmersson anders.helmersson@liu.se