Fundamental problems in fault detection and identification

Ali Saberi¹, Anton A. Stoorvogel², Peddapullaiah Sannuti³ and Henrik Niemann⁴,*†

¹School of Electrical Engineering, Washington State University, Pullman, WA 99164-2752, U.S.A.
²Department of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
³Department of Electrical and Computer Engineering, Rutgers University, 94 Brett Road, Piscataway, NJ 08854-8058, U.S.A.
⁴Department of Automation, Building 326, Technical University of Denmark, DK-2800 Lyngby, Denmark

SUMMARY
A number of different fundamental problems in fault detection and fault identification are formulated in this paper. The fundamental problems include exact, almost, generic and class-wise fault detection and identification. Necessary and sufficient conditions for the solvability of the fundamental problems are derived. These conditions are weaker than the ones found in the literature since we do not assume any particular structure for the residual generator. At the end of the paper, a time domain synthesis procedure based on state-space methods to construct appropriate residual generators is given. Copyright © 2000 John Wiley & Sons, Ltd.

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1. INTRODUCTION
Various types of faults arise in industrial processes owing to malfunction of internal components of a process as well as those in measurement sensors and control actuators attached to the process. Over the last three or four decades industrial automation has been increasingly fueled by various technological developments including the availability of highly complex electronic equipment and the overwhelming progress in computer technology. This has led not only to the development of complex control systems but also to higher demand for reliable and secure control systems. Thus, it has become imperative that any fault that occurs is detected and identified automatically without severely disturbing the yield the process generates. This has stimulated over the last two decades an extensive study of fault detection and identification methods.

As discussed in a survey paper by Willsky [1], one faces three different types of tasks or layers in the area of fault detection and identification, (1) fault detection, (2) fault identification, and

*Correspondence to: Henrik Niemann, Department of Automation, Building 326, Technical University of Denmark, DK-2800 Lyngby, Denmark.
†E-mail: hhn@iau.dtu.dk

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(3) fault estimation. **Fault detection** consists of designing a residual generator that produces a residual signal enabling one to make a binary decision as to whether a fault occurred or not. **Fault identification** imposes a stronger requirement. When one or more faults occur, the residual signal must enable us not only to detect that there are faults occurring in the system, but it must also enable us to identify (isolate) which faults have occurred. Finally, **fault estimation** is the determination of the extent of failure. The latter is done by trying to reconstruct the fault signals.

A number of fundamental problems that arise in fault estimation, i.e. in estimating the fault signals have been studied recently by us [2]. For both continuous- as well as discrete-time systems, in addition to so-called exact estimation of fault signals, the notion of almost estimation has been introduced. Exact estimation of a fault signal requires that the transfer function from the fault signal to its estimate be an identity matrix, while the transfer function from any disturbance to the estimate of the fault signal is identically zero. In contrast to this, almost estimation of a fault signal requires that the transfer function from the fault to its estimate be as close as desired to the identity matrix in a certain norm ($H_2$ or $H_\infty$), while a chosen norm ($H_2$ or $H_\infty$) of the transfer function from any disturbance to the estimate of the fault signal be as small as required. The notion of almost estimation, as opposed to the exact estimation, weakens the required solvability conditions considerably. This is not surprising to the readers familiar with control theory literature on exact and almost disturbance decoupling. Furthermore, for discrete-time systems, another significant notion of using a fixed delay in estimating a fault signal has been introduced in Reference [2]. That is, at time step $k$, one obtains the estimate of the fault signal at $k - \ell$ where $\ell$ is a fixed non-negative integer. Again, the motivation to introduce such a delay arises from the desire to weaken the required solvability conditions. Besides finding solvability conditions to the various fault estimation problems, the paper [2] also presents constructive state space methods to find these fault estimators.

In this paper, we consider fault detection and identification without basing it on fault signal estimation. Here we focus our attention on certain fundamental problems of fault detection and identification that were originally formulated in a very clear way in the work by Massoumnia et al. [3] who provided solvability conditions for the problems they formulated. However, the work of Massoumnia et al. is concerned only with actuator faults. Sensor faults are transformed into actuator faults by considering the sensor faults as the outputs of an, arbitrarily chosen, strictly proper dynamic system. As a consequence of this, the solvability conditions for the formulated problems of fault detection and identification would naturally depend on the selected dynamic system. Obviously, such results are not attractive from a theoretical point of view. Other papers followed [3] and utilized approaches such as unknown input observers, eigenstructure and frequency domain methods [3–11]. All of these papers consider exact fault detection and/or fault identification problems which impose that the residual signal which is generated is unaffected by disturbance inputs, while the transfer function matrix from the fault signal vector to the residual signal vector has a specific structure to allow fault detection and identification. In other words, exact fault detection and identification is a problem of designing a residual generator on the basis of available measurement of the process so that (1) the resulting residual signal is exactly decoupled from the disturbance, and (2) the transfer function matrix from the fault signal vector to the residual signal vector has a certain structure, typically a diagonal structure. Furthermore, in some of the above methods, the solvability conditions depend on the apriori selected residual generator architecture, e.g. residual generator based on unknown input observers, etc.

Here we redefine the traditional problems of fault detection and identification and also introduce new ones. The motivation for new definitions and problem formulations arises from
a desire to make them practically meaningful with a potential to weaken their solvability conditions. To do so, we have injected or incorporated three new notions into our problem formulations. Essentially, these notions are (1) almost fault detection and identification, (2) generic fault detection and identification, and (3) class-wise fault identification.

The notion of almost fault detection and identification is motivated by the vast control literature that exists on exact and almost disturbance decoupling. In almost fault detection and identification, we seek in a natural way an *almost* disturbance decoupling of the residual signal rather than the *exact* disturbance decoupling. That is, we require that the transfer function matrix from the disturbance to the residual signal be arbitrarily small in either $H_2$ or $H_\infty$ norm sense rather than requiring it to be exactly zero. We observe that the weakening of the *exact* disturbance decoupling requirement to an *almost* disturbance decoupling requirement makes great sense from an engineering point of view. As mentioned earlier, such an almost fault detection and identification based on fault signal estimation was considered recently by us in Reference [2] in connection with fault estimation. It turned out in Reference [2] that the solvability conditions for almost fault estimation are weaker than the solvability conditions for exact fault estimation. However, in this paper, we focus on fault detection and identification instead of fault estimation and we will see that for fault detection and identification we will not be able to weaken the solvability conditions by only asking for *almost* fault detection or identification instead of *exact* detection or identification.

The notion of generic fault detection and identification is based on excluding certain occurrences of fault signals that have exact static or dynamic relationships with each other. That is, in some cases whenever fault signals satisfy among themselves certain exact static or dynamic relationships, faults cannot be detected or identified. It is obviously highly meaningful to exclude such cases because the occurrence of multiple faults having exact relationships among fault signals has a zero probability. Fault detection and identification problems based on the exclusion of such occurrences of fault signals with exact relationships between them is termed as generic fault detection and identification problems. In these problems, one seeks fault detection and identification in almost all cases but for a very special subclass of faults which are almost surely not occurring in practice. A class of such problems where certain static relationships among fault signals is excluded was first introduced in Reference [3]. The genericity introduced in this paper is much broader as it considers dynamic as well as static relationships among fault signals.

The third notion we introduce here relates to the situation whenever individual fault identification is not possible, i.e. the solvability conditions for individual fault identifiability are not met. In this case, we can introduce the notion of classwise fault identification which attempts to identify in what subclass of all possible faults, the fault or faults occurred. For this purpose, we divide all possible faults into mutually disjoint subclasses $\Gamma_i, i = 1, 2, \ldots, v$. Having done so, one can then seek whether a fault has occurred in a particular subclass or not rather than seeking individual fault identification. Obviously, such a classwise fault identification could result in a weakening of the required solvability conditions.

By utilizing the above three notions, we will formulate in Section 2 fourteen different problems of fault detection and identification, and then study these problems in order to establish the necessary and sufficient conditions under which they can be solved. Following that, whenever such conditions are satisfied, a time-domain synthesis procedure will be developed in Section 3 to construct appropriate residual generators. It needs to be pointed out that the solvability conditions given in this paper are independent of the apriori selected type of residual generator architecture.
As discussed above, utilization of the notions of almost, generic, and class wise fault detection or fault identification weakens some of the requirements. Another direction of weakening the problem requirements is delayed fault detection and identification. That is, one could allow making a decision regarding the detection and identification of faults for time \( k - \ell \), using the information available up to \( k \) where \( \ell \) is a certain fixed time delay. However, such a time delayed fault detection or fault identification can only be studied in detail when we incorporate the time it takes to detect or identify into the problem statements. In all the problems we formulate in this paper, we state that if a component of the residual vector \( r \) becomes non-zero then a fault has occurred in the past. But there is no restriction or measure imposed on the time between occurrence and detection or identification of a fault. Therefore, allowing delays does not help at all with respect to weakening the solvability conditions of any problem.

2. PROBLEM FORMULATIONS AND MAIN RESULTS

Consider the following state-space description for a plant or a system given by

\[
\Sigma: \begin{cases} 
\sigma x = Ax + \sum_{j=1}^{m} E_j d_j + \sum_{i=1}^{k} L_i f_i \\
= Ax + Ed + L_f f \\
y = Cx + \sum_{j=1}^{m} D_{d,j} d_j + \sum_{i=1}^{k} D_{f,i} f_i \\
= Cx + D_d d + D_f f, 
\end{cases}
\]

where \( \sigma \) is an operator indicating the time derivation \( d/dt \) for continuous-time systems and a forward unit time shift for discrete-time systems. Also, \( x \in \mathbb{R}^n \) is the state vector, \( d \in \mathbb{R}^m \) is a disturbance signal vector, and \( y \in \mathbb{R}^p \) is the measurement vector. Furthermore, \( f_i \) signifies the \( i \)th fault for each \( i = 1, 2, \ldots, k \). The coefficient matrices \( L_i \) and \( D_{f,i} \) are referred to in the literature as failure signatures associated with the \( i \)th fault, while \( f_i \) itself is called the \( i \)th fault signal. Obviously, the failure signatures \( L_i \) and \( D_{f,i} \) depend on the physics of the given system. The fault signal vector \( f \in \mathbb{R}^k \) is a collection of fault signals \( f_i, i = 1, 2, \ldots, k \), into a vector. We will sometimes need to refer to the fault signal \( \tilde{f}_i \in \mathbb{R}^k \) which is a vector with all elements equal to zero except for the \( i \)th position where it is equal to \( f_i \). Because there is no possibility for confusion and to simplify notation we will denote both \( \tilde{f}_i \) and \( f_i \) by \( f_i \). It is always clear from the context which interpretation we are using. The above system can be rewritten in a transfer function form as

\[
y = G_d d + G_f f
\]

In modelling a given plant by system (1), we assume that all the fault signals \( f_i, i = 1, 2, \ldots, k \), are quite arbitrary and that no information is known regarding their characteristics. That is, none of the signals \( i = 1, 2, \ldots, k \), are constrained to belong to any special class of functions. We now proceed to formulate certain fault detection and identification problems. The fault detection setup we follow here is shown in Figure 1.

Let the residual signal \( r \) be given by

\[
r = Hy = \Psi(d, f)
\]
where \( r \) is a time function that takes values in \( \mathbb{R}^q \). In general, we might have to take \( H \) to be a nonlinear bounded-input, bounded-output stable operator which makes \( \Psi \) also a non-linear operator mapping disturbances and faults to a residual signal \( r \). Of course, if \( H \) is linear, then there exist transfer matrices \( G_{rf} \) and \( G_{rd} \) such that

\[
r = G_{rf}f + G_{rd}d
\]

One of the basic issues that concerns fault detection and identification is whether one can achieve such a detection and identification when the disturbance \( d \) affects the system. This points out a need to have a residual generator which is insensitive to the external disturbance \( d \). That is, we need that

\[
\Psi(d, f) = \Psi(0, f)
\]

for all disturbances \( d \) and all fault signals \( f \) or at least that the dependence of \( r \) on \( d \) is arbitrarily small with respect to some specified norm. If \( H \) is linear then this implies that we impose that the transfer matrix \( G_{rd} \) is zero or arbitrarily small in some specific norm.

Before we proceed, we need to consider certain modelling aspects. In a given situation, there exists always a number of possible faults. Some of these individual faults might occur simultaneously at any given time and others cannot. The tasks of fault detection and identification depend on which faults can occur simultaneously and which cannot. Based on the information available as to what faults could occur simultaneously at any time and what cannot, one divides the set of all possible faults into mutually exclusive and exhaustive sets. To do so, let us introduce some notation. Let us denote the set of all possible faults by \( k = \{1, 2, \ldots, k\} \). Based on the known information, let \( k \) be partitioned into \( \ell \) mutually exclusive and exhaustive sets, \( \Omega_i, i = 1, 2, \ldots, \ell \). That is, let \( \Omega_i \cap \Omega_j = \emptyset \) for \( i \neq j \), and \( \Omega_1 \cup \Omega_2 \cup \cdots \Omega_{\ell} = k \). Also, let \( k_i \) denote the number of elements in \( \Omega_i \). This leads us to define the following simultaneous occurrence property.

**Simultaneous occurrence property:** Only those faults that belong to any single set among the sets \( \Omega_i, i = 1, 2, \ldots, \ell \), can occur simultaneously at any given time. This implies that certain faults belonging to a set say \( \Omega_i \) and others that belong to a set say \( \Omega_j, i \neq j \), cannot occur simultaneously at any given time.

Two special and extreme cases of the general simultaneous occurrence property are interesting and important. The first extreme case where \( \ell = 1 \) is called **simultaneous occurrence property of type 1**. In this case, all possible \( k \) faults can occur simultaneously at any given time. This case is interesting, important, and probably the most natural one as it requires the least information as to what faults can occur simultaneously at any given time since it includes the case when any two
or more faults can occur simultaneously at any given time. If one can solve the problem of identification or detection for faults of type 1, one does not need to consider situations involving any other types. On the other hand, modeling all the possible faults into a type other than type 1 requires more information, and hence presumably the task of identification and detection in that case is simpler than the same task for faults of type 1. The other extreme case of simultaneous occurrence property corresponds to the case when $\ell = k$, and is called simultaneous occurrence property of type 2. In this case, each and every fault occurs by itself, i.e. it never occurs simultaneously with any other fault, and as such it is interesting and important.

The Section 2.1 defines the basic problems that correspond to fault detection, the Section 2.2 defines the basic problems that correspond to fault identification, and finally the Section 2.3 develops the solvability conditions for all the problems we formulated.

2.1. Fault detection

As mentioned in the introduction, the task of fault detection consists of designing a residual generator that produces a residual signal enabling one to make a binary decision as to whether a fault or faults occurred or not. In this subsection, precise fault detection problem statements are developed by first considering whether a certain single fault, say the $i$th fault, is detectable or not. In order to formulate formally the detectability of any single fault, we rewrite below the system given in (1) under the assumption that there exists a single fault:

$$
\dot{x} = Ax + \sum_{j=1}^{m} E_j d_j + L_i f_i \\
y = Cx + \sum_{j=1}^{m} D_{d,j} d_j + D_{f,i} f_i
$$

We have the following problem formulation.

**Problem 1**

Suppose there exists a single fault, say the $i$th fault. Then, for the system given in (3) the problem of (exact) fault detection of a single fault with signatures $L_i$ and $D_{f,i}$ is finding, if existent, a bounded-input–bounded-output stable residual generator $H_i$ whose output is a scalar residual signal $r_i = \Psi_i(d, f_i)$ such that

1. $\Psi_i(d, 0) = 0$ for all disturbances $d$.
2. $\Psi_i(d, f_i) \neq 0$ for all $f_i \neq 0$ and all disturbances $d$.

We say that the $i$th fault with signatures $L_i$ and $D_{f,i}$ is exactly detectable if the above problem is solvable.

Obviously, if the residual generator is linear, then we impose that $H_i \in \mathcal{RH}_\infty$ such that $G_{rd} = 0$ and $G_{rf_i}$ is non-zero.

We note that fault detection imposes an exact decoupling of residual signal $r$ from the disturbance $d$. Motivated by the existing control theory literature on disturbance decoupling, we expect then that the solvability conditions for this problem will be severe. Again, guided by the literature on almost disturbance decoupling, seemingly we can weaken such solvability conditions by imposing merely that the disturbance $d$ influence the residual signal $r$ as small as desired rather than having no influence at all. With this in mind, we have the following problem:
Problem 2
Suppose there exists a single fault, say the $i^{th}$ fault. Then, for the system given in (3) the problem of almost fault detection of a single fault with signatures $L_i$ and $D_{f,i}$ is finding, if existent, a $\delta > 0$ and a parameterized family of bounded-input–bounded-output stable residual generators $H_{i,e}$ whose output is a scalar residual signal $r_i = \Psi_{i,e}(d, f_i)$ such that for all $\varepsilon > 0$,

1. $\|\Psi_{i,e}(d, 0)\|_2 \leq \varepsilon \|d\|_2$ for all disturbances $d$.
2. $\|\Psi_{i,e}(d, f_i)\|_2 \geq \delta \|f_i\|_2$ for all disturbances $d$ and all fault signals $f_i$.

The $i^{th}$ fault with signatures $L_i$ and $D_{f,i}$ is said to be almost detectable if the problem of almost fault detection is solvable.

Obviously, if the parameterized family of residual generators is linear, then we need to impose that $H_{i,e} \in \mathcal{H}_\infty$ is such that $\|G_{r_i,f_i,e}\|_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$ and $\|G_{r_i,f_i,e}\|_\infty \geq \delta$ for all $\varepsilon > 0$.

Problems 1 and 2 discuss the detectability of a single fault. An immediate question that arises is what can one do if a fault is not detectable. Obviously, one needs additional information in order that it can be detected. For instance, one might have additional information that the fault signal amplitude is large in comparison with the amplitude of the disturbances, or that the frequency content of the fault signal is different from that of the disturbances, and so on. Utilizing such additional information in fault detection is indeed important. However, we will not pursue this line of research in this paper but pursue it in subsequent papers.

The above discussion pertains to detectability of single faults. We now consider the case when multiple faults are possible. Of course, whenever multiple faults are possible, one faces not only the task of fault detection but also the task of identification. There are two versions of detectability of multiple faults that need to be considered. The following problem concerns with the first version, and essentially imposes that the residual signal be insensitive to the disturbances, and for any arbitrary fault signal $f$ unequal to zero the resulting residual signal be unequal to zero.

Problem 3
Consider the system given in (1) under the simultaneous occurrence property. The problem of (exact) fault detection of a set of multiple faults $f$ with signature matrices $L_f$ and $D_f$ is defined as the problem of finding, if existent, a bounded-input–bounded-output stable residual generator $H$ whose output is a scalar residual signal $r = \Psi(d, f)$ such that

1. $\Psi(d, 0) = 0$ for all disturbances $d$.
2. $\Psi(d, f) \neq 0$ for all faults $f \neq 0$ and all disturbances $d$.

We say that the set of multiple faults with signature matrices $L_f$ and $D_f$ is exactly detectable if the above problem is solvable for it.

In the linear case, when all faults can occur simultaneously, this requires the transfer matrix $G_{rf}$ from $f$ to $r$ to be left-invertible and the transfer matrix $G_{rd}$ from $d$ to $r$ to be identically 0.

On the other hand, the second version involves a weaker formulation by introducing the notion of genericity into problem formulation. A certain kind of static genericity was first introduced in [3]. The notion of static genericity used in [3] can be illustrated by the following example:

$$
\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\
y = (1 \ 1) x
$$
where \( f_1 \) and \( f_2 \) are two possible faults. Fault detection is not possible for this system because, if \( f \) is non-zero such that \( f_2 = -f_1 \), then we can get the same measurement as \( f = 0 \). However, \( f_2 = -f_1 \) basically requires the two different faults \( f_1 \) and \( f_2 \) to generate exactly identical but opposite fault signals which will never happen in practice. In [3], a condition is imposed so that \( r \neq 0 \) for all non-zero fault signals which do not satisfy a static linear relationship.

The notion of genericity we utilize here is much broader and goes beyond the static relationship utilized in [3] and covers dynamic relationships as well. To illustrate this, we consider the following example:

\[
\begin{align*}
\dot{x} &= f_1 \\
y &= x + f_2
\end{align*}
\]

There exist non-zero fault signals satisfying \( f_2 = -f_1 \) which generate zero measurements and which can hence not be detected. There is no static linear relationship between the non-zero fault signals which generate zero measurements, and hence the fault is not detectable according to the definition in [3]. However, there is a dynamic relationship \( f_2 = -f_1 \) between the faults that generate zero measurements, and this dynamic relationship seems to be equally unlikely as a static relationship between the fault signals. Therefore, the weakest condition which still seems to detect all faults that one can reasonably expect in practice is that the transfer matrix from \( f_i \) to \( r \) is unequal to zero for all faults, and the faults are independent in the sense that the fault signatures are independent meaning that

\[
\begin{pmatrix}
L_f \\
D_f
\end{pmatrix}
\]

is left-invertible. This implies that the only way we might miss detection is that more than one fault occurs at the same time and in such a way that their effect on \( r \) just happens to cancel against each other via a linear static or dynamic relationship. Generic detection will be connected to a genericity matrix which is a square transfer matrix with \( k \) columns which are all unequal to zero. In view of this discussion, we introduce the following problem of generic fault detection.

**Problem 4**

Consider the system given in (1) under the simultaneous occurrence property. The problem of generic fault detection of a set of multiple faults \( f \) with independent signature matrices \( L_f \) and \( D_f \) is defined as a problem of finding, if existent, a bounded-input–bounded-output stable residual generator whose output is a scalar residual signal \( r = \Psi(d, f) \) such that there exists a genericity matrix \( V \) and such that

1. \( \Psi(d, 0) = 0 \) for all disturbances \( d \),
2. \( \Psi(d, f) \neq 0 \) for all disturbances \( d \) and for all fault signals \( f \) such that \( Vf \neq 0 \).

We say that the set of multiple faults with signature matrices \( L_f \) and \( D_f \) is generically detectable if the above problem is solvable for it.

For linear residual generators the above conditions are equivalent to the requirement that the transfer matrix \( G_{rd} \) from \( d \) to \( r \) is zero and the transfer matrix \( G_{rf_i} \), from fault \( f_i \) to \( r \) is non-zero for all \( i = 1, \ldots, k \).

As we shall see in Section 2.3, the notion of generic fault detection introduced above leads to an important conclusion, namely that a set of faults with signature matrices \( L_f \) and \( D_f \) is generically detectable.
detectable if and only if each individual fault is detectable and the fault signatures are independent. This means that the generic detectability of a vector fault can equivalently be arrived at by examining the detectability of each and every individual component of the vector fault.

We revisit now Problems 3 and 4, and weaken the requirement of exact insensitivity to disturbances by requiring only almost insensitivity, i.e. we inject now the notion of almost fault detection.

Problem 5
Consider the system given in (1) under the simultaneous occurrence property. The problem of almost fault detection of a set of multiple faults $f$ with signature matrices $L_f$ and $D_f$ is defined as a problem of finding, if existent, a $\delta > 0$ and a parameterized family of bounded-input–bounded-output stable residual generators $H_e$ whose output is a scalar residual signal $r = \Psi_e(d, f)$ such that for all $\varepsilon > 0$,

1. $\|\Psi_e(d, 0)\|_2 \leq \varepsilon \|d\|_2$ for all disturbances $d$.
2. $\|\Psi_e(d, f)\|_2 \geq \delta \|f\|_2$ for all disturbances $d$ and all fault signals $f$.

We say that the set of multiple faults with signature matrices $L_f$ and $D_f$ is almost detectable if the above problem is solvable for it.

Obviously, we can also formulate a generic version of the above problem, and it is given below.

Problem 6
Consider the system given in (1) under the simultaneous occurrence property. The problem of generic almost fault detection of a set of multiple faults $f$ with signature matrices $L_f$ and $D_f$ is defined as a problem of finding, if existent, a $\delta > 0$, a genericity matrix $V$, and a parameterized family of bounded-input–bounded-output stable residual generators $H_e$ whose output is a scalar residual signal $r = \Psi_e(d, f)$ such that for all $\varepsilon > 0$,

1. $\|\Psi_e(d, 0)\|_2 \leq \varepsilon \|d\|_2$ for all disturbances $d$.
2. $\|\Psi_e(d, f)\|_2 \geq \delta \|Vf\|_2$ for all disturbances $d$ and all fault signals $f$.

We say that the set of multiple faults with signature matrices $L_f$ and $D_f$ is generically almost detectable if the above problem is solvable for it.

2.2 Fault identification
The task of fault detection exists only when there is a possibility of multiple faults occurring. In that case, in addition to detecting that a fault or faults occurred, one has to identify as to what individual fault or faults have occurred. Let us also emphasize that fault identification can be sought at different levels. The most demanding level is the one that seeks to identify each and every individual fault that occurred. In certain aspects of engineering, one may not need to identify each and every individual fault that occurred. Perhaps, one can classify all possible faults into certain classes. Then, one needs simply to ascertain that a fault or faults belonging to a particular class or classes have occurred.

In connection with fault identification, it is important to recognize that all possible faults are at least generically detectable to start with, since there is no sense talking about fault identification for faults that are not at least generically detectable. Thus, the detectability of a set of faults must be ascertained before one faces the task of fault identification.
We first consider the case when we need to identify each and every individual fault that occurred. It is easy now to recognize that the task of isolating or identifying every individual fault requires that we generate for each individual fault signal $f_i$ a dedicated residual signal $r_i$ such that $r_i$ would be insensitive to all disturbances and all vector faults for which $f_i$ is identical to zero while it is sensitive to all vector faults for which $f_i$ is not identical to zero. It is clear from this discussion that, for the task of individual fault identification, the dimension of residual vector can always be taken the same as the dimension of fault vector $f$ itself.

We can now have the following precise formulation of fault identification problem that seeks to identify each individual fault that occurred.

**Problem 7**

Consider the system given in (1) under the simultaneous occurrence property. Then, the problem of (exact) individual fault identification for a set of faults $f$ with signature matrices $L_f$ and $D_f$ is defined as a problem of finding, if existent, a bounded-input–bounded-output stable residual generator $H$ which generates a residual vector $r = \Psi(d, f)$ such that for any fault $f_i$, $i = 1, 2, \ldots, k$, there exists a dedicated component $r_i$ of $r$ and the operator $\Psi_i$ from $d$ and $f$ to $r_i$ has the following properties:

- $\Psi_i(d, f) = 0$ for any disturbance $d$ and any fault $f$ such that $f_i$ is identical to zero.
- $\Psi_i(d, f) \neq 0$ for any disturbance $d$ and any fault $f$ such that $f_i$ is not identical to zero.

The set of faults $f$ with signature matrices $L_f$ and $D_f$ is said to be individually identifiable if the problem of individual fault identification is solvable.

Evidently, a necessary condition for (exact) individual fault identification is fault detection. Although the problem formulation is independent of the fact whether or not certain faults can occur simultaneously, we will see that the solvability condition will obviously depend on this additional information.

Note that one can ask whether there is a possibility to weaken the solvability conditions by imposing only generic individual fault identification. As we will see later, this is not the case if we have a simultaneous occurrence property of type 1 (where all faults can occur simultaneously). However, in general the solvability conditions might be different if we only require generic individual fault identification. To illustrate some issues consider the following example:

\[
\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} f
\]

\[
y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} f
\]

with $\Omega_1 = \{1, 2\}$, $\Omega_2 = \{3\}$, and $\Omega_3 = \{4\}$. For this case the fault identification is generically solvable as can be most easily seen in the frequency domain,

\[
y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} f_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f_2 + \begin{pmatrix} 1 \\ s+1 \end{pmatrix} f_3 + \begin{pmatrix} 1 \\ s+1 \end{pmatrix} f_4
\]

Assume we apply the following identification scheme:

- If $[1/(s + 1)] y_1 = y_2$, then fault $f_4$ has occurred,
• If \( y_1 = \frac{1}{s+1} y_2 \), then fault \( f_3 \) has occurred,
• If \( y_1 \neq \frac{1}{s+1} y_2 \), and \( \frac{1}{s+1} y_1 \neq y_2 \), then \( f_1 \) has occurred if \( y_1 \neq 0 \) and \( f_2 \) has occurred if \( y_2 \neq 0 \).

This scheme identifies all faults correctly but for two non-generic cases:

• If \( \frac{1}{s+1} f_1 = f_2 \) \( (f_2 \neq 0) \), then we identify fault \( f_3 \) which is obviously incorrect.
• If \( f_1 = \frac{1}{s+1} f_2 \) \( (f_2 \neq 0) \), then we identify fault \( f_4 \) which is again incorrect.

The two cases for which we obtain an incorrect result are obviously highly unlikely. Two faults have to occur simultaneously and their fault signals have to satisfy a precise relationship to obtain an incorrect result. Therefore we still would like to say that in this case the generic individual fault identification problem is solvable.

In fault detection we defined a genericity matrix (a square transfer matrix with nonzero columns) and defined that the result is generic if there exists one genericity matrix \( V \) such that the fault detection scheme detects all faults except possibly for those faults that satisfy \( Vf = 0 \). In the above example, we see that for fault identification this result is not satisfactory. In this case, there should exist at most a finite number of genericity matrices, \( V_1, \ldots, V_s \), such that we identify all faults except for possibly those faults for which there exists an \( i \in \{1, \ldots, s\} \) for which \( V_if = 0 \).

We need more than one genericity matrix because we have seen in the example that there exists two cases where we obtain an incorrect result and we cannot capture these two cases in one genericity matrix. For the above example we need two genericity matrices,

\[
V_1 = \begin{pmatrix} 1 & 1 \\ 0 & s+1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 1 \\ s+1 & 0 \end{pmatrix}
\]

In general, it can be shown that we need no more than \( \ell - 1 \) genericity matrices. In particular, for the case \( \ell = 1 \) (simultaneous occurrence property of type 1), we do not need any genericity matrices and therefore generic individual fault identification is solvable if and only if (exact) individual fault identification is solvable.

In view of the above introduction, let us formally define generic individual fault identification as follows.

**Problem 8**

Consider the system given in (1) under the simultaneous occurrence property. Then, the problem of **generic individual fault identification** for a set of faults \( f \) with signature matrices \( L_f \) and \( D_f \) is defined as a problem of finding, if existent, a bounded-input–bounded-output stable residual generator \( H \) which generates a residual vector \( r = \Psi(d, f) \) such that there exists a finite number of genericity matrices, \( V_1, \ldots, V_s \), such that for any fault \( f_i \), \( i = 1, 2, \ldots, k \), there exists a dedicated component \( r_i \) of \( r \) and the operator \( \Psi_i \) from \( d \) and \( f \) to \( r_i \) has the following properties:

• \( \Psi_i(d, 0) = 0 \) for any disturbance \( d \),
• \( \Psi_i(d, f) = 0 \) for any disturbance \( d \) and for any fault \( f \) such that \( V_j f \neq 0 \) for all \( j = 1, \ldots, s \), and such that \( f_i \) is identical to zero,
• \( \Psi_i(d, f) \neq 0 \) for any disturbance \( d \) and any fault \( f \) such that \( V_j f \neq 0 \) for all \( j = 1, \ldots, s \), and \( f_i \) is not identical to zero.
The set of faults $f$ with signature matrices $L_f$ and $D_f$ is said to be individually generically identifiable if the problem of generic individual fault identification is solvable.

We note that the problems of exact and generic individual fault identification (like the problem of fault detection) impose a decoupling of residual vector $r$ from the disturbance $d$. As we did in the case of fault detection, we can again weaken this requirement by requiring that the conditions are satisfied arbitrarily well but not perfectly.

**Problem 9**

Consider the system given in (1) under the simultaneous occurrence property. Then, the problem of almost individual fault identification for a set of faults $f$ with signature matrices $L_f$ and $D_f$ is defined as a problem of finding, if existent, a $\delta > 0$ and a parameterized family of bounded-input–bounded-output stable residual generators $H_e$ which generates a residual vector $r = \Psi_e(d, f)$ such that for any $\epsilon > 0$ and for any fault $f_i$, $i = 1, 2, \ldots, k$, there exists a dedicated component $r_i$ of $r$ and the operator $\Psi_{i,e}$ from $d$ and $f$ to $r_i$ has the following properties:

- For any disturbance $d$ and any fault $f$ such that $f_i$ is identical to zero
  \[ \|\Psi_{i,e}(d, f)\|_2 \leq \epsilon \|d\|_2 + \epsilon \|f\|_2 \]

- For any disturbance $d$ and any fault $f$ such that $f_i$ is not identical to zero,
  \[ \|\Psi_{i,e}(d, f)\|_2 \geq \delta \|f_i\|_2 - \epsilon \|d\|_2 - \sum_{j=1, j \neq i}^{s} \epsilon \|f_j\|_2 \]

The set of faults $f$ with signature matrices $L_f$ and $D_f$ is said to be almost individually identifiable if the above problem of almost individual fault identification is solvable.

We have the following generic version of the above problem.

**Problem 10**

Consider the system given in (2) under the simultaneous occurrence property. Then, the problem of generic almost individual fault identification for a set of faults $f$ with signature matrices $L_f$ and $D_f$ is defined as a problem of finding, if existent, a $\delta > 0$ and a parameterized family of bounded-input–bounded-output stable residual generators $H_e$ which generates a residual vector $r = \Psi_e(d, f)$ such that there exists a finite number of genericity matrices, $V_1, \ldots, V_s$, such that for any $\epsilon > 0$ and for any fault $f_i$, $i = 1, 2, \ldots, k$, there exists a dedicated component $r_i$ of $r$ and the operator $\Psi_{i,e}$ from $d$ and $f$ to $r_i$ has the following properties:

- For any disturbance $d$
  \[ \|\Psi_{i,e}(d, 0)\|_2 \leq \epsilon \|d\|_2 \]

- For any disturbance $d$ and any fault $f$ such that $f_i$ is identical to zero
  \[ \min_{j=1, \ldots, s} \|V_j f\|_2 \|\Psi_{i,e}(d, f)\|_2 \leq \epsilon \|f\|_2 (\|d\|_2 + \|f\|_2) \]

- For any disturbance $d$ and any fault $f$ such that $f_i$ is not identical to zero,
  \[ \|\Psi_{i,e}(d, f)\|_2 \geq \min_{j=1, \ldots, s} \delta \|V_j f\|_2 - \epsilon \|d\|_2 - \sum_{k=1, k \neq i}^{s} \epsilon \|f_k\|_2 \]
The set of faults $f$ with signature matrices $L_f$ and $D_f$ is said to be almost individually generically identifiable if the above problem of generic almost individual fault identification is solvable.

Evidently, a necessary condition for almost individual fault identification is almost fault detection. Although the problem formulation is independent of the fact whether or not certain faults can occur simultaneously, we will see that the solvability condition will obviously depend on this additional information.

As we said earlier, the case of simultaneous occurrence property of type 1 is probably the most commonly used one. It is natural to do so because it requires no information at all regarding what faults can occur simultaneously and what cannot. In this case a natural question arises especially when one or both of the fault identification problems formulated above are not solvable for the given model (1). What would be the largest subset of faults that is (exactly) individually identifiable? The following problem is concerned with this aspect.

**Problem 11**

Consider the system given in (1) under the simultaneous occurrence property of type 1. Assume that the given set of faults $f$ with signature matrices $L_f$ and $D_f$ is not individually identifiable. Then, the problem is to obtain the largest subset of faults $f_s$ such that it is (exactly) individually identifiable.

The following problem is analogous to the above problem but is concerned with almost individual fault identification.

**Problem 12**

Consider the system given in (1) under the simultaneous occurrence property of type 1. Assume that the given set of faults $f$ with signature matrices $L_f$ and $D_f$ is not almost individually identifiable. Then, the problem is to obtain the largest subset of faults $f_s$ such that it is almost individually identifiable.

So far we have discussed the identifiability of all individual faults that can occur. Thus, if faults, say $f_{i_1}, f_{i_2},$ and $f_{i_3}$, that belong to a particular set $\Omega_i$ have occurred simultaneously at any given time, we sought to identify each individual fault $f_{i_1}, f_{i_2},$ and $f_{i_3}$. On the other hand, in certain engineering applications, one may not need such a detailed individual fault identification. All one needs is perhaps to identify that, say, a fault or faults in actuator occurred, or a fault or faults in a sensor occurred, or in general a fault or faults in a particular specified part of the plant occurred.

In order to identify such classes of faults, we can partition all possible faults into several mutually exclusive and exhaustive classes, $\Gamma_i, i = 1, 2, \ldots, v$. That is, let $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, and $\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_v = \Gamma$. Then, we need to identify whether a fault or faults occurred in one or more of these classes. In other words, if faults say $f_{i_1}, f_{i_2},$ and $f_{i_3}$, that belong to a particular set $\Omega_i$ have occurred simultaneously at any given time, and let us say $f_{i_1}$ is an element of $\Gamma_\alpha$ and $f_{i_2},$ and $f_{i_3}$ are elements of $\Gamma_\beta$, then we simply need to identify that faults in classes $\Gamma_\alpha$ and $\Gamma_\beta$ occurred without individually isolating $f_{i_1}, f_{i_2},$ and $f_{i_3}$.

It is easy to see for the purpose of isolating classes of faults rather than isolating individual faults, one needs one dedicated residual signal for each class of faults. That is, for any $i$, $i = 1, 2, \ldots, v$, a single residual signal $r_i$ needs to be dedicated to all faults that belong to the class $\Gamma_i$. This implies that the dimension of residual vector $r$ needs to be $v$ rather than $k$. In this situation, $r_i$ is sensitive to all faults that belong to the class $\Gamma_i$ but insensitive to all other faults and to all disturbances.
To illustrate the concept of classwise fault identification, as an example, let us consider faults \( f_1, f_2, \) and \( f_3 \) all belonging to the set \( \Omega \) have occurred simultaneously at any given time. Let us say \( f_1 \) is an element of \( \Gamma_x \) and \( f_2, f_3 \) are elements of \( \Gamma_y \). Then, in this case, \( r_x \) being sensitive to \( f_1 \) gets excited while \( r_y \) being sensitive to \( f_2 \) gets excited and \( f_3 \) gets also excited; thus enabling us to identify that both classes \( \Gamma_x \) and \( \Gamma_y \) are at fault.

If we have a simultaneous occurrence property of type 1 then it can be shown that classwise identification with complete detection is solvable only if we have individual fault identification. If individual fault identification is possible, then there is obviously no need for a concept like classwise identification. This is no longer the case with different types of simultaneous occurrence. However the results for classwise identification then become quite technical while we believe that the generic case is any way the most relevant problem. Therefore, we will consider here only generic classwise identification in combination with generic detection. We recall that generic detection can be connected to a finite number of genericity matrices which are square transfer matrix with \( k \) non-zero columns.

We can now have the following precise formulation of generic classwise fault identification problem that seeks to identify what class or classes of faults occurred.

**Problem 13**

Consider the system given in (1) under the simultaneous occurrence property. Then, the problem of generic classwise fault identification for a set of detectable faults \( f \) with signature matrices \( L_f \) and \( D_f \) is defined as a problem of finding, if existent, a bounded-input–bounded-output stable residual generator \( H \) which generates a residual vector \( r = \Psi(d,f) \) such that there exist a finite number of genericity matrices, \( V_1, \ldots, V_s \), such that for any class of faults \( \Gamma_x, x = 1, 2, \ldots, v \), there exists a dedicated component \( r_x \) of \( r \) and the operator \( \Psi_x \) from \( d \) and \( f \) to \( r_x \) has the following properties:

- \( \Psi_x(d, 0) = 0 \) for any disturbance \( d \).
- \( \Psi_x(d, f) = 0 \) for any disturbance \( d \) and any fault \( f \) such that \( V_j f \) unequal to zero for all \( j = 1, \ldots, s \), and such that \( f_i \) is identical to zero for all \( i \in \Gamma_x \).
- \( \Psi_x(d, f) \neq 0 \) for any disturbance \( d \) and any fault \( f \) such that we have \( V_j f \) unequal to zero for all \( j = 1, \ldots, s \), and such that \( f_i \) is unequal to zero for at least one \( i \in \Gamma_x \).

The set of faults \( f \) with signature matrices \( L_f \) and \( D_f \) is said to be classwise identifiable if the problem of classwise fault identification is solvable.

Evidently, Problem 13 has been formulated for generically detectable faults since it is insane to talk about fault identification of a set of faults if any individual fault in the set is not generically detectable. We note that the satisfaction of all the conditions in Problem 13 guarantees that every fault in the set \( f \) is detectable. However, for a given set of faults, if there exists some faults which are not detectable, the conditions of Problem 13 are immediately violated.

We note that the problem of classwise fault identification (like the problem of fault detection and the problem of individual fault identification) imposes a decoupling of residual vector \( r \) from the disturbance \( d \). As we did in almost fault detection and almost individual fault identification, we can again weaken this requirement by requiring that the conditions are satisfied arbitrarily well but not perfectly.

**Problem 14**

Consider the system given in (1) under the simultaneous occurrence property. Then, the problem of generic almost classwise fault identification for a set of detectable faults \( f \) with signature
matrices $L_f$ and $D_f$ is defined as a problem of finding, if existent, a $\delta > 0$, a finite number of genericity matrices, $V_1, \ldots, V_s$, and a parameterized family of bounded-input–bounded-output stable residual generators $H_e$ which generates a residual vector $r = \Psi_e(d, f)$ such that for any class of faults $\Gamma_x, x = 1, 2, \ldots, v$, there exists a dedicated component $r_x$ of $r$ and for all $\varepsilon > 0$ the operator $\Psi_{x, \varepsilon}$ from $d$ and $f$ to $r_x$ has the following properties:

- For any disturbance $d$,
  \[ \|\Psi_{x, \varepsilon}(d, 0)\|_2 \leq \varepsilon \|d\|_2 \]

- For any disturbance $d$ and any fault $f$ such that $f_i$ is identical to zero for all $i \in \Gamma_x$,
  \[ \min_{j=1, \ldots, s} \|V_j f\|_2 \|\Psi_{x, \varepsilon}(d, f)\|_2 \leq \varepsilon \|f\|_2 (\|d\|_2 + \|f\|_2) \]

- For any disturbance $d$ and any fault $f$ such that $f_i$ is not identical to zero for some $i \in \Gamma_x$, we have
  \[ \|\Psi_{x, \varepsilon}(d, f)\|_2 \geq \delta \min_{j=1, \ldots, s} \|V_j f\|_2 - \sum_{i \in l_x} e \|f_i\|_2 - \varepsilon \|d\|_2 \]

The set of faults $f$ with signature matrices $L_f$ and $D_f$ is said to be almost class-wise identifiable if the problem of almost classwise fault identification is solvable.

Again, it is evident that Problem 14 has been formulated for almost detectable faults since it is insane to talk about almost classwise fault identification of a set of faults if any individual fault in the set is not almost detectable. We note that the satisfaction of all the conditions in Problem 14 guarantees that every fault in the set $f$ is almost detectable. However, for a given set of faults, if there exists some faults which are not almost detectable, the conditions of Problem 14 are immediately violated.

2.3. Solvability conditions

So far we have formulated a number of problems concerned with fault detection and isolation. In this subsection, we proceed to develop the necessary and sufficient conditions under which such problems are solvable. As we shall see later on, the conditions we develop not only show when the formulated problems are solvable, but also form road maps to construct appropriate residual generators that can be implemented.

All the solvability conditions developed here are stated in terms of the normrank of certain transfer matrices. Hence, let us recall that $G$ denotes the normal rank of the transfer matrix $G$, i.e. the rank of $G(s)$ for all $s \in \mathbb{C}$ but a finitely many points.

2.3.1. Solvability conditions for detection problems. We first study exact and almost fault detection of a single fault as defined in Problems 1 and 2. Our study in this regard leads to the result given below in Theorem 2.1. As we said in introduction, surprisingly, the solvability conditions for both the exact and almost detection problems turn out to be the same.
Theorem 2.1
Consider the system given (3). Then, the following statements hold:

1. The problem of (exact) fault detection of a single fault with signatures $L_i$ and $D_{f,i}$ is solvable if and only if
   \[
   \text{normrank } (G_{yd} G_{sf,i}) > \text{normrank } G_{yd}
   \]  

2. The problem of almost fault detection of a single fault with signatures $L_i$ and $D_{f,i}$ is solvable if and only if the problem of fault detection is solvable as formulated in Problem 1.

Moreover, whenever the normrank condition given in (4) is satisfied, one can construct a linear residual generator that solves the exact or almost fault detection of a single fault.

Proof. The proof of this theorem follows immediately from the proof of the next theorem which deals with the problem of fault detection of multiple faults.

We now move on to study the detection of multiple faults. In this regard, we have four different problems related to exact, almost, generic, and almost generic fault detection as formulated in Problems 3–6. The study of all these problems needs to be done under the broad notion of simultaneous occurrence property. Theorem 2.2 presents the results of our study. In presenting these results and elsewhere, we denote by $\#\Omega_s$ the number of elements in the set $\Omega_s$ and by $f_{1s}$ the subset of all faults in $\Omega_s$.

Theorem 2.2
Consider the system given in (1) under the simultaneous occurrence property. Then, the following statements hold:

1. The problem of (exact) fault detection with signature matrices $L_f$ and $D_f$ is solvable if and only if
   \[
   \text{normrank } (G_{yd} G_{sf}) \geq \text{normrank } G_{yd} + \#\Omega_s \quad \text{for } \alpha = 1, \ldots, \ell
   \]  

2. The problem of almost fault detection with signature matrices $L_f$ and $D_f$ is solvable if and only if the problem of fault detection is solvable.

3. The problem of generic fault detection with signature matrices $L_f$ and $D_f$ is solvable if and only if
   \[
   \text{normrank } (G_{yd} G_{sf,i}) > \text{normrank } G_{yd} \quad \text{for all } i = 1, \ldots, k.
   \]  

4. The problem of generic almost fault detection with signature matrices $L_f$ and $D_f$ is solvable if and only if the problem of generic fault detection is solvable.

Note that, if all faults can occur simultaneously (simultaneous occurrence property of type 1), then (5) reduces to

\[
\text{normrank } (G_{yd} G_{sf}) \geq \text{normrank } G_{yd} + k.
\]

Proof. Obviously, there exists a stable rational matrix $W$ with a stable inverse such that

\[
WG_{yd} = \begin{pmatrix} G_{d1} \\ 0 \end{pmatrix}, \quad WG_{sf} = \begin{pmatrix} G_{f1} \\ G_{f2} \end{pmatrix}
\]
with $G_{d_1}$ right invertible. By $G_{f_0}$ we denote the transfer matrix consisting of the columns of $G_{f_2}$ associated to $\Omega_s$. We can then find that
\[
\text{normrank } (G_{yd} G_{yf}) = \text{normrank } G_{d_1} + \text{normrank } G_{f_2} = \text{normrank } G_{yd} + \text{normrank } G_{f_2}
\] (8)
where we used the right-invertibility of $G_{d_1}$.

Now assume we have a residual generator $H$ which solves Problem 3. We choose a scalar $a \in \mathbb{R}$ such that $a G_{d_1}$ has a stable right inverse which we will denote by $(a G_{d_1})^{-1}$. Our residual generator must also achieve fault detection of $f_1$ in the special case,
\[
f = a f_1
\]
\[
d = (a G_{d_1})^{-1} G_{f_1} f_1,
\]
but in this case
\[
y_1 \equiv 0
\]
\[
y_2 = a G_{f_2} f_1
\]
Assume $G_{f_0,2}$ is not injective. In that case there exists a $f_1 \neq 0$ (with only the elements associated to $\Omega_s$ unequal to zero), such that $y_2 \equiv 0$. But then the residual generator cannot distinguish the non-zero fault $f = a f_1$ (which is associated to $\Omega_s$) with disturbance $d$ from the case of no faults and zero disturbances (since both yield $y \equiv 0$). The latter yields a contradiction with the fact that we were able to achieve fault detection. Hence a necessary condition for fault detection is
\[
\text{normrank } G_{f_0,2} \geq \# \Omega_s
\]
Combined with (8) this yields (5).

Similarly, assume that we can achieve almost fault detection, and condition (5) is not satisfied. Assume that for each $\varepsilon$ the residual signal is given by $r = \Psi_{s}(d, f)$. Then we have just seen that there exists a non-zero fault $f$ and a disturbance $d$ such that
\[
y = G_{yd} d + G_{yf} f = 0
\]
But then $\Psi_{s}(d, f) = \Psi_{s}(0, 0)$. Almost fault detection guarantees $\Psi_{s}(0, 0) = 0$ but then
\[
\|\Psi_{s}(d, f)\|_2 = 0 < \delta \|f\|_2
\]
for any $\delta > 0$ which contradicts almost fault detection.

Conversely if (5) is satisfied, we must have that $G_{f_0,2}$ has full column rank for $a = 1, \ldots, \ell$, and hence the choice
\[
H = (0 \ 1) W
\]
achieves a transfer matrix $G_{yd} = 0$ and a transfer matrix $G_{yf}$ with the property that the submatrix of all columns associated to $\Omega_s$ has full column rank for all $a = 1, \ldots, \ell$. Hence, if faults occur in $\Omega_s$, then $r = G_{yf} f \neq 0$ for all $a = 1, \ldots, \ell$. However, in fault detection we required a scalar residual signal. We can obtain this by using a nonlinear mapping $s: \mathbb{R}^k \rightarrow \mathbb{R}$ such that $s(v) \neq 0$ for all $v \in \mathbb{R}^k, v \neq 0$ and $s(0) = 0$. One particular choice for $s$ is given by $s(v) = \|v\|$. Then it is immediate that $s \circ H$ solves the fault detection problem. This complete the proof of parts 1 and 2 of the theorem.

Next, note that for generic fault detection it is clearly necessary that each individual fault can be detected (assuming the other faults cannot occur). This immediately applies, according to part 1,
that (6) must be satisfied for all $i = 1, \ldots, k$. For almost generic fault detection it is obvious from part 2 that in this case also (6) must be satisfied for all $i = 1, \ldots, k$.

Conversely, if (6) is satisfied for all $i = 1, \ldots, k$, then there exists according to part 1 a residual generator for each $i = 1, \ldots, k$ which solves the fault detection problem for each $f_i$. In the proof of part 1 we have seen that we can choose the residual generator of one single fault linear. If we concatenate these $k$ residual generators, we obtain a stable transfer matrix $H$ such that for $r = Hy$ we obtain

$$r = G_{rf}f + G_{rd}d$$

The fact that the $i$th element of the vector $r$ is a fault detector for fault $f_i$ implies that $G_{rd}$ must be equal to zero and all the diagonal elements of the transfer matrix $G_{rf}$ must be non-zero. It is then obvious that there exists a rowvector $v \in \mathbb{R}^k$ such that $vG_{rf}$ is a row vector which has only nonzero elements. This is, for instance, guaranteed if $v^T$ is such that it is orthogonal to any of the columns of $G_{rf}$. Since none of the columns is equal to zero, the set of rowvectors which are not orthogonal to any column of $G_{rf}$ clearly contains nonzero elements. Finally, it is straightforward to check that the residual generator $vH$ solves the generic fault detection problem. $\square$

**Remark 2.1**

A close examination of the proof of Theorem 2.2 reveals the structure of a residual generator that solves the posed problems. Evidently, the residual generator has the structure of a linear system followed by a static non-linearity.

**Remark 2.2**

As we anticipated, the above theorem shows that under the framework of our notion of genericity, if all individual faults are exactly detectable, then a vector fault is generically detectable whenever all the fault signatures are independent. Hence, the notion of generic detectability of a vector fault becomes essentially equivalent to the detectability of each and every individual component of the vector fault.

### 2.3.2. Solvability conditions for identification problems

In this subsection, we study different identification problems.

We first proceed to present our results on the problems of exact and almost individual fault identification as defined in Problems 7 and 9. We denote by $\#\Omega_s$ the number of elements in the set $\Omega_s$ and by $\Omega_{x,y} \bigcup \Omega_{y,x}$ the subset of all faults in $\Omega_{x,y} \bigcup \Omega_{y,x}$. By $G_{yf_s}$ we denote the transfer matrix from $f_s$ to $y$ which is obtained from $G_{rf}$ by retaining only the columns in $\Omega_s \bigcup \Omega_{y,x}$.

**Theorem 2.3**

Consider the system given (1) under the simultaneous occurrence property. Then, the following statements hold:

1. The problem of (exact) individual fault identification for a set of faults $f$ which satisfy the simultaneous occurrence property and which together have the fault signature matrices $L_f$ and $D_f$ is solvable if and only if the following condition is true:

$$\text{normrank} \left( G_{yd} \quad G_{yf_s} \right) \geq G_{yd} + \#\Omega_s + \#\Omega_{y,x}$$

for all $s$, $\beta \in \{1, \ldots, f'\}$ with $s \neq \beta$.
2. The problem of *almost individual fault identification* for a set of faults $f$ which satisfy the simultaneous occurrence property and which together have the fault signature matrices $L_f$ and $D_f$ is solvable if and only if the problem of *individual fault identification* for the set $f$ is solvable.

Note that condition (9) is indeed a stronger condition than the solvability condition for the fault detection problem as given in (5) which is consistent with the obvious conclusion that fault identification is a stronger requirement than fault detection.

**Proof.** For any $\Omega_x$ and $\Omega_y$ where $\alpha \neq \beta$, the problem of fault identification requires that the problem of fault detection be solvable for the set of all faults in $\Omega_x \cup \Omega_y$ (under the condition that all faults from $\Omega_x \cup \Omega_y$ can occur simultaneously). This can be seen by noting from the proof of Theorem 2.2 that, if such a fault detection cannot be achieved, then there exist a particular non-zero fault signal $f$ which is a combination of nonzero faults from $\Omega_x \cup \Omega_y$,

$$\sum_{i \in \Omega_x} a_i f_i + \sum_{j \in \Omega_y} b_j f_j$$

and a particular disturbance $d$ which yields a zero measurement signal $y$. But then, we cannot distinguish between the fault

$$\sum_{i \in \Omega_x} a_i f_i$$

with disturbance $d$ and the fault

$$-\sum_{j \in \Omega_y} b_j f_j$$

with disturbance 0 since linearity of the system implies that in both cases we obtain the same measurement signal $y$. This is in contradiction with our assumption that fault identification is possible. This implies that fault detection for signals from $\Omega_x \cup \Omega_y$ must be possible (under the condition that all faults from $\Omega_x \cup \Omega_y$ can occur simultaneously). By Theorem 2.2 we find that (9) must be satisfied.

It remains to prove the converse. Assume that (9) is satisfied. Then we use a similar decomposition as in the proof of Theorem 2.2. There exists a stable rational matrix $W$ with a stable inverse such that

$$WG_{yd} = \begin{pmatrix} G_{d1} \\ 0 \end{pmatrix}, \quad WG_{yf_{x,y}} = \begin{pmatrix} G_{f_{x,y1}} \\ G_{f_{x,y2}} \end{pmatrix}$$

with $G_{d1}$ right invertible where the transfer matrix $G_{yf_{x,y}}$ is constructed by extracting the columns $\Omega_x \cup \Omega_y$ from $G_{yf}$. Hence the choice $r = \bar{H} y$ with

$$\bar{H} = (0 \ I) W$$

in view of (9) achieves a transfer matrix $G_{rf_{x,y}}$ which has full column rank and a transfer matrix $G_{rd} = 0$. Let $S$ denote a left-inverse of $G_{rf_{x,y}}$. Then $SH$ is a residual generator that achieves $G_{rd} = 0$ and $G_{rf_{x,y}} = I$. However, $SH$ will in general not be stable. For each row of $SH$, say $(SH)_i$, there obviously exists a stable rational function $g_i$ such that $g_i(SH)_i$ is stable and non-zero. Let

$$\Omega_x = \{ \alpha_1, \ldots, \alpha_p \}, \quad \Omega_y = \{ \beta_1, \ldots, \beta_q \}$$
Next we consider the following residual generator,

\[
H_{\alpha\beta} = \begin{pmatrix}
g_1 & 0 & \cdots & 0 \\
0 & g_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & g_{p_s+p_f}
\end{pmatrix} S H
\]

If we define

\[
\begin{pmatrix} r^\alpha_{\alpha,\beta} \\
r^\beta_{\alpha,\beta}
\end{pmatrix} = r_{\alpha,\beta} = H_{\alpha,\beta} y
\]

(where \(r^\alpha_{\alpha,\beta}\) takes values in \(\mathbb{R}^{p_s}\) and \(r^\beta_{\alpha,\beta}\) takes values in \(\mathbb{R}^{p_f}\), then we know that if \(r^\alpha_{\alpha,\beta}\) is unequal to zero then either a fault in \(\Omega_\alpha\) has occurred or a fault outside of \(\Omega_\alpha \cup \Omega_\beta\) has occurred. Owing to the simultaneous occurrence property, it is not possible that a fault in \(\Omega_\beta\) has occurred. If we combine all the residual generators \(H_{\alpha\beta}\) then we can detect in which set the fault has occurred, that is

- If \(r^\alpha_{\alpha,\beta} = 0\) for at least one \(\alpha \neq \beta\), then no fault has occurred in \(\Omega_\alpha\).
- If \(r^\alpha_{\alpha,\beta}\) is unequal to zero for all \(\alpha \neq \beta\), then a fault has occurred in \(\Omega_\alpha\).

If a fault has occurred in \(\Omega_\alpha\), then we can detect whichever fault has occurred in it. More specifically, consider a fault in \(\Omega_\alpha\), say \(f_\alpha\), for \(\alpha_i \in \Omega_\alpha\). If there exists a \(\beta \neq \alpha\) such that \(r^\alpha_{\alpha,\beta} = 0\), then \(f_\alpha\) has not occurred. On the other hand, if \(r^\alpha_{\alpha,\beta} \neq 0\), and \(r^\alpha_{\alpha,\gamma} \neq 0\) for \(\gamma \neq \alpha, \beta\), then \(f_\alpha\) has occurred. In other words, if we define \(r_\alpha\) for \(\alpha_i \in \Omega_\alpha\) as

\[
r_\alpha = r^\alpha_{\alpha,\beta} \prod_{\gamma \neq \alpha, \beta} \| r^\alpha_{\alpha,\gamma} \|
\]

with \(\beta \neq \alpha\) chosen arbitrarily, then we build a residual vector \(r\) with the property that \(r_\alpha \neq 0\) if and only if fault \(f_\alpha\) has occurred. In other words, we can actually achieve fault identification.

The above theorem leads to the following corollary:

**Corollary 2.4**

Consider the system (1) under the simultaneous occurrence property of type 1. The total number of faults that can be identified while solving either exact or almost fault identification is equal to

\[
\text{normrank } (G_{yd} G_{sf}) - \text{normrank } G_{yd}\tag{10}
\]

The total number of faults that can be detected and identified given in Corollary 4, need not be the total number of faults in the system. In the case when a limited number of faults occur simultaneously or if we allow for generic detection then we can potentially identify an arbitrarily large number of faults.

We next look at the solvability conditions for generic (almost) individual fault identification which obviously yield weaker solvability conditions.
Theorem 2.5

Consider the system given (1) under the simultaneous occurrence property. Then, the following statements hold:

1. The problem of generic individual fault identification for a set of faults $f$ which satisfy the simultaneous occurrence property and which together have the fault signature matrices $L_f$ and $D_f$ is solvable if and only if the following condition is true: For any $i = 1, \ldots, k$ with $\pi$ such that $i \in \Omega_\pi$, we have

$$\text{normrank}(G_{yd} G_{yf_i} G_{yf_{\pi \setminus i}}) > \text{normrank}(G_{yd} G_{yf_{\pi \setminus \{i\}}})$$  \hspace{1cm} (11)

where $f_{\pi \setminus i}$ is the subset of faults in $\Omega_\pi$ excluding $f_i$. Moreover, for all $\beta \neq \pi$ we have

$$\text{normrank}(G_{yd} G_{yf_i} G_{yf_{\beta \setminus i}}) > \text{normrank}(G_{yd} G_{yf_{\beta \setminus \{i\}}})$$  \hspace{1cm} (12)

2. The problem of generic almost individual fault identification for a set of faults $f$ which satisfy the simultaneous occurrence property and which together have the fault signature matrices $L_f$ and $D_f$ is solvable if and only if the problem of generic individual fault identification for the set $f$ is solvable.

Proof. Let $f_i \in \Omega_\pi$ be given. Assume that there exist a particular non-zero fault signal $f$ which is a combination of non-zero faults from $\{i\} \cup \Omega_\pi$ with $\beta \neq \pi$,

$$f_i + \sum_{j \in \Omega_\pi} b_j f_j,$$  \hspace{1cm} (13)

and a particular disturbance $d$ which yields a zero measurement signal $y$. Then obviously, if fault $f_i$ has occurred, one cannot identify this correctly because $f_i$ with disturbance $d$ and $- \sum_{j \in \Omega_\pi} b_j f_j$ with disturbance 0 generate the same measurement signal. Generic individual fault identification however requires that when only one fault occurs, then it should always be correctly identified. Using the same arguments as in the proof of Theorem 2.2, it is then easily seen that if (13) with disturbance $d$ can never generate a zero measurement signal then we must have that (12) is satisfied.

Similarly, if there exists a particular non-zero fault signal $f$ which is a combination of non-zero faults from $\Omega_\pi$,

$$f_i + \sum_{j \in \Omega_\pi \setminus \{i\}} b_j f_j,$$

and a particular disturbance $d$ which yields a zero measurement signal $y$, then we again have a contradiction and this implies that (11) must be satisfied.

It remains to prove the converse. Assume that (12) is satisfied. For any $i$ with $\pi$ such that $i \in \Omega_\pi$, we again use a similar decomposition as in the proof of Theorem 2.2. By (12), there exists a stable rational matrix $W$ with a stable inverse such that

$$WG_{yd} = \begin{pmatrix} G_{d_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad W(G_{yf_i} G_{yf_{\pi \setminus i}}) = \begin{pmatrix} G_{f_i,1} & G_{yf_{\pi \setminus \{i\},1}} \\ G_{f_i,2} & G_{yf_{\pi \setminus \{i\},2}} \\ G_{f_i,3} & \mathbf{0} \end{pmatrix}$$
with $G_{d_1}$ right invertible and $G_{f_3}$ a non-zero scalar transfer matrix. Hence the choice $r_{i,\Omega_3} = H_{i,\Omega_3} y$ with

\[
H_{i,\Omega_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

obtains the residual signals

\[
\begin{pmatrix} r_{i,\Omega_3} \\ r_{i,\Omega_3} \end{pmatrix} = r_{i,\Omega_3} = H_{i,\Omega_3} y
\]

with $r_{i,\Omega_3}$ scalar. If we combine all the residual generators $H_{i,\Omega_3}$, then we can generically detect which faults have occurred, that is

- If there exists an $i \in \Omega_3$ for which $r_{i,\Omega_3}$ is unequal to zero for all $\beta \neq \alpha$, then a fault in $\Omega_3$ has occurred.
- If for all $i \in \Omega_3$ there exists $\beta$ for which $r_{i,\Omega_3}$ is equal to zero, then no fault in $\Omega_3$ has occurred.

The question in generic fault identification is to see when the above mechanism yields a wrong conclusion. The conclusion that a fault in $\Omega_3$ has occurred is always correct. However, we might incorrectly conclude that no fault in $\Omega_3$ has occurred. If a fault $f_\alpha$ has occurred, then we fail to detect it, if for each $i \in \Omega_3$ there exists a $\beta_i$ such that

\[
0 = r_{i,\Omega_3} = H_{i,\Omega_3} y = G_{r_i,y_i} f_{\alpha_i}
\]

But it is easy to check that the transfer matrix

\[
V = \begin{pmatrix} G_{r_i,y_i,f_{\alpha_i}} \\ \vdots \\ G_{r_{\alpha_i},y_{\alpha_i},f_{\alpha_i}} \end{pmatrix}
\]

has non-zero columns and hence is a genericity matrix so the only errors that we make are due to non-generic faults. It is also obvious that there are only a finite number of these genericity matrices.

If a fault has occurred in $\Omega_3$, then we still need to detect which faults in $\Omega_3$ have occurred. Here we use (11). Consider a fault signal $f_i$ with $i \in \Omega_3$. There exists a stable rational matrix $W$ with a stable inverse such that

\[
WG_{y_i} = \begin{pmatrix} G_{d_1} \\ 0 \\ 0 \end{pmatrix}, \quad W(G_{j,y_i} G_{y_f,\alpha}) = \begin{pmatrix} G_{f_1,1} & G_{y_f,\alpha_1} \\ G_{f_1,2} & G_{y_f,\alpha_2} \\ G_{f_1,3} & 0 \end{pmatrix}
\]

with $G_{d_1}$ right invertible and $G_{f_3}$ a non-zero scalar transfer matrix. Hence the choice $r_{i,\Omega_3} = H_{i,\Omega_3} y$ with

\[
H_{i,\Omega_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
obtains the residual signals,

\[
\begin{pmatrix}
    r_{i,\Omega_x}^{\text{rest}} \\
    r_{i,\Omega_x}^d \\
    r_{i,\Omega_x}
\end{pmatrix} = r_{i,\Omega_x} = H_{i,\Omega_x} y
\]

with \( r_{i,\Omega_x} \) being a scalar. If we combine all the residual generators, \( H_{i,\Omega_x} \), then with the \textit{a priori} information that a fault has occurred in \( \Omega_x \), we can generically detect which faults in \( \Omega_x \) have occurred, that is

- If for \( i \in \Omega_x \) we have that \( r_{i,x} \) is unequal to zero, then \( f_i \) has occurred.
- If for \( i \in \Omega_x \) we have that \( r_{i,x} \) is equal to zero, then \( f_i \) did not occur.

The question in generic fault identification is to see when the above mechanism yields a wrong conclusion. We see that we cannot make a wrong conclusion, assuming of course that our \textit{a priori} information that a fault has occurred in \( \Omega_x \) is correct.

By combining the above arguments it is then easy to construct a suitable residual generator. It remains to show that the solvability conditions cannot be weakened if we allow for an almost formulation. As we have seen in the beginning of this proof, if the solvability conditions are not satisfied, then there exists a single fault \( f_i \) which cannot be correctly identified since it generates the same measurement signal as other faults given suitable disturbance signals. It is then obvious that we cannot correctly identify this fault in an almost sense either because with identical measurement signals there is no information available to make an appropriate choice.

\( \square \)

\textbf{Remark 2.3}

In the above proof the number of genericity matrices which might lead to incorrectly identified faults is quite high. It is rather technical but it can be shown via a more detailed analysis that we never need more than \( k - 1 \) genericity matrices.

\textbf{Remark 2.4}

We would like to emphasize the significance of the above theorems. They clearly show that if almost fault detection or almost fault detection and identification is possible, then exact fault detection or exact fault detection and identification is possible as well. This means that for the purpose of only fault detection and identification (i.e. if we are not interested in an estimate of the fault signal), an almost formulation does not relax the solvability condition from that of the exact formulation. This \textbf{does} not necessarily make the almost fault detection or identification problems irrelevant. In fact, as shown in [2], if one wants to estimate or reconstruct the fault signal, then one would face different solvability conditions for exact and almost fault detection or identification problems.

The following theorem gives the solvability conditions for the classwise fault identification problem. We need some notation. For any \( a \in \{1, \ldots, v\} \), let \( f_i \) be an element of \( \Gamma_a \), and \( x \) be such that \( f_i \) be an element of \( \Omega_x \) as well. Thus, \( f_i \in \Gamma_a \cap \Omega_x \). Next, let \( f_{i,\beta} \) denote the subset of faults in \( (\Omega_x \cup \Omega_\beta) \cap \Gamma_a \) where \( \Gamma_a \) denotes the faults which are not in \( \Gamma_a \). By \( G_{yf_{i,\beta}} \) we denote the transfer matrix from \( f_{i,\beta} \) to \( y \) which is obtained from \( G_{yf_i} \) by retaining only the columns in \( \Omega_\beta \cap \Gamma_a \).
Theorem 2.6

Consider the system given (1) under the simultaneous occurrence property. Then, the following statements hold:

1. The problem of *classwise fault identification* for a set of faults $f$ which satisfy the simultaneous occurrence property and which together have the fault signature matrices $L_f$ and $D_f$ is solvable if and only if the following condition is true: For any $i = 1, \ldots, k$ and any $\beta \in \{1, \ldots, \ell\}$, we have

$$\text{normrank} (G_{yd} G_{yf_i}, G_{yf_{1:\ell}}) > \text{normrank} (G_{yd} G_{yf_{1:\ell}})$$

(14)

where $f_{i, \beta}$ is the subset of faults in $\Omega_{i, \beta} \cap \Gamma_\alpha$.

2. The problem of *almost classwise fault identification* for a set of faults $f$ which satisfy the simultaneous occurrence property and which together have the fault signature matrices $L_f$ and $D_f$ is solvable if and only if the problem of *classwise fault identification* for the set $f$ is solvable.

Proof. Let $f_i \in \Omega_{i, \beta} \cap \Gamma_\alpha$ be given. Assume that there exist a particular non-zero fault signal $f$ which is a combination of non-zero faults from $\Gamma^* \cap \Omega_\beta$,

$$f_i = \sum_{j \in \Gamma^* \cap \Omega_\beta} b_j f_j$$

(15)

and a particular disturbance $d$ which yields a zero measurement signal $y$. Then obviously, if fault $f_i$ has occurred, one cannot identify this correctly because $f_i$ with disturbance $d$ and $- \sum_{j \in \Gamma^* \cap \Omega_\beta} b_j f_j$ with disturbance $0$ generate the same measurement signal. Generic identification however requires that when only one fault occurs then it should always be correctly identified. Using the same arguments as in the proofs of Theorems 2.2 and 2.5, it is then easily seen that, if (15) with disturbance $d$ can never generate a zero measurement signal, then we must have that (14) is satisfied.

It remains to prove the converse. Assume that (14) is satisfied. We again use a similar decomposition as in the proof of Theorem 2.5. By (14), for any $i = 1, \ldots, k$, and any $\beta \in \{1, \ldots, \ell\}$, there exists a stable rational matrix $W$ with a stable inverse such that

$$WG_{yd} = \begin{pmatrix} G_{f_1} \\ 0 \\ 0 \end{pmatrix}, \quad W(G_{yf_i}, G_{yf_{1:\ell}}) = \begin{pmatrix} G_{f_{i,1}} & G_{yf_{1:1}} \\ G_{f_{i,2}} & G_{yf_{1:2}} \\ G_{f_{i,3}} & 0 \end{pmatrix}$$

with $G_{f_1}$ right invertible and $G_{f_{i,3}}$ a non-zero scalar transfer matrix. Hence the choice $r_{i, \Omega_\beta} = H_{i, \Omega_\beta} y$ with

$$H_{i, \Omega_\beta} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix} W$$

obtains the residual signals,

$$\begin{pmatrix} r_{i, \Omega_\beta} \\ r_{i, \Omega_\beta} \end{pmatrix} = H_{i, \Omega_\beta} y$$
with \( r^j_{i,\Omega_b} \) being a scalar. If we combine all the residual generators \( H_{i,\Omega_b} \), then we can generically detect in which set the fault has occurred, that is

- If there exists an \( i \in \Gamma_a \) for which \( r^j_{i,\Omega_b} \) is unequal to zero for all \( \beta \in \{1, \ldots, \ell \} \), then a fault in \( \Gamma_a \) has occurred.
- If for all \( i \in \Gamma_a \) there exists a \( \beta \) for which \( r^j_{i,\Omega_b} \) is equal to zero, then no fault in \( \Gamma_a \) has occurred.

The question in generic fault identification is to see when the above mechanism yields a wrong conclusion. The conclusion that a fault in \( \Omega_x \) has occurred is always correct. However, we might incorrectly conclude that no fault in \( \Omega_x \) has occurred. If a fault \( f_{\Omega_a} \) has occurred, then we fail to detect it if for each \( i \in \Omega_x \) there exists a \( \beta_i \) such that

\[
0 = r^j_{i,\Omega_b} = H_{i,\Omega_b} y := G_{r_i,\Omega_i f_{i_m}}.
\]

But it is easy to check that the transfer matrix

\[
V = \begin{pmatrix} G_{r_1,\beta_1,\Omega_2} \\ \vdots \\ G_{r_{\ell},\beta_{\ell},\Omega_2} \end{pmatrix}
\]

has non-zero columns and hence is a genericity matrix, so the only errors that we make are due to non-generic faults. It is also obvious that there are only a finite number of these genericity matrices.

By combining the above arguments it is then easy to construct a suitable residual generator. It remains to show that the solvability conditions cannot be weakened if we allow for an almost formulation. As we have seen in the beginning of this proof, if the solvability conditions are not satisfied, then there exists a single fault \( f_i \) in \( \Gamma_a \) which cannot be correctly identified since it generates the same measurement signal as other faults outside of \( \Gamma_a \) given suitable disturbance signals. It is then obvious that we cannot correctly identify this fault in an almost sense either because with identical measurement signals there is no information available to make an appropriate choice.

\[ \square \]

3. TIME-DOMAIN DESIGN OF RESIDUAL GENERATOR

We would like to state an important aspect of our results here. The proofs of Theorems 2.2 and 2.3 and the ones that follow them are constructive, and therefore yield suitable residual generators for all the problems as defined in this paper. Basically, there are two components which play crucial roles in the construction of residual generators. The fundamental components in the proofs is the construction of an appropriate stable rational matrix \( W \) with a stable inverse and having certain properties. That is, given \( y = G_{y_d} d + G_{y_f} f \), one needs to find a stable transfer matrix \( W \) with

\[
WG_{y_d} = \begin{pmatrix} G_{d1} \\ 0 \end{pmatrix}, \quad WG_{y_f} = \begin{pmatrix} G_{f1} \\ G_{f2} \end{pmatrix},
\]

and with \( G_{d1} \) right invertible. (Sometimes variations occur but from a construction point of view the above is the central question.) Although the existence of such a matrix \( W \) is obvious as mentioned in the proofs, a method of constructing such a matrix was not discussed in the proofs. Another fundamental issue one needs to clarify relates to the construction of a stable transfer
matrix $S$ such that $SG_{f2}$ is diagonal with non-zero elements on the diagonal in case we have $G_{f2}$ left-invertible. These two fundamental issues of construction are discussed below. In fact, we present construction methods in time domain which are reliable and yields lower order residual generators, in particular the number of zeros at the origin being minimized.

Before we provide the construction methods for appropriate $W$ and $S$, we need to recall the definition of a geometric subspace which plays a crucial role in our construction.

**Definition 3.1**
Consider a linear system $\Sigma$ characterized by the quadruple $(A, B, C, D)$.

The detectable strongly controllable subspace $S^d(\Sigma)$ is defined as the minimal subspace of $\mathbb{R}^n$ which is $(A + KC)$ invariant and contains $\text{Im}(B + KD)$ such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\mathbb{R}^n/S^d$ are contained in $C_g \subseteq \mathbb{C}$ for some $K$.

In the case of continuous-time systems, $C_g$ will represent the open left-half complex plane $\mathbb{C}^-$. In the case of discrete-time systems, $C_g$ will represent the set of complex numbers inside the unit circle.

We have the following system,

$$\Sigma: \begin{cases} \sigma x = Ax + Ed + L_{f2} \\ y = Cx + D_d d + D_{f2} \end{cases}$$

Assume for simplicity that the system is stable. Note that designing a residual generator for the above system is intrinsically the same as designing a residual generator for the following system,

$$\Sigma: \begin{cases} \sigma x = Ax + Ed + L_{f2} + Ky \\ y = Cx + D_d d + D_{f2} \end{cases}$$

Actually this new system $y = \tilde{G}_{yd} d + \tilde{G}_{f2} f$ is constructed from the old one via the transformation $\tilde{G}_{yd} = WG_{yd}$, $\tilde{G}_{f2} = WG_{s2}$ with $W$ a stable transfer matrix with stable inverse defined by $W(s) = C(sI - A)^{-1}K + I$. Now we choose $K$ such that

$$(A + KC)S^d(\Sigma_1) \subseteq S^d(\Sigma_1)$$

$$\text{Im}(E + KD_d) \subseteq S^d(\Sigma_1)$$

$$S^d(\Sigma_1) |_{\mathbb{R}^n/S^d(\Sigma_1)} \text{ is stable}$$

where $\Sigma_1$ is the system characterized by the quadruple $(A, E, C, D_d)$. Next we decompose the state space $\mathcal{X} = S^d(\Sigma_1) \oplus \mathcal{X}_2$, and the measurement space $\mathcal{Y} = (C_1 S^d(\Sigma_1) + \text{Im} D_d) \oplus \mathcal{Y}_2$. In this new basis, the system has the form

$$\Sigma: \begin{cases} \sigma x = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} x + \begin{pmatrix} E_1 \\ 0 \end{pmatrix} d + \begin{pmatrix} L_{f1} \\ L_{f2} \end{pmatrix} f \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix} x + \begin{pmatrix} D_{d1} \\ 0 \end{pmatrix} d + \begin{pmatrix} D_{f1} \\ D_{f2} \end{pmatrix} f \end{cases}$$

with $A_{22}$ stable. Since the transfer matrix from $d$ to $y_2$ is left-invertible, it is easy to show that we basically achieved the structure as required in decomposition (15).
Next, we need to find a stable transfer matrix $S$ such that $SG_{f_2}$ is diagonal with non-zero diagonal elements. We have a state-space realization for $G_{f_2}$ which is of lower order than the original system,

$$
\Sigma_{f_2}: \begin{cases}
\sigma x_2 = A_{22} x_2 + L_{f_2} f \\
y_2 = C_{22} x_2 + D_{f_2} f
\end{cases}
$$

Next, assume that the system $\Sigma_{f_2}$ is left-invertible. For this system we now execute a minimum-phase/all-pass factorization as described in detail in [12]. We obtain transfer matrices $G_m$ and $W$ (both with corresponding state-space representations) such that $G_{f_2} = G_m W$ where $G_{f_2}$ is the transfer matrix of the system $\Sigma_{f_2}$. We know that $G_m$ is minimum-phase and stable while $W$ is square, stable and inner. Now we can design a residual generator on the basis of the transfer matrix of the system $\Sigma_{f_2}$ such that $y_n = W f$ since we can trivially construct $y_n$ from $y_2$ since $G_m$ is minimum-phase and left-invertible. The latter follows from the fact that we know that $\Sigma$ must be left-invertible.

Now, the system with transfer matrix $W$ is obviously invertible, and because the system is inner, the inverse has transfer matrix $W^*$ defined by $W^*(s) = W^T(-s)$. If we now choose $S = W^*G_m^L$, then $SG_{f_2}$ is the identity matrix where $G_m^L$ is a stable left inverse of $G_m$. The only remaining problem is the fact that $W^*$ is not stable.

We will find a diagonal matrix $W_d$ such that $W_d W^*$ is stable. Let $W(s) = C_v(sI - A_v)^{-1} B_v + D_v$. This will be done by row of $W^*$. Denote by $v_i$ the $i$th row of $W^*$ with $v_i(s) = C_{v,i}(sI - A_v)^{-1} B_v + D_{v,i}$. We construct some matrix $H$ such that $A_v + HC_{v,i}$ is stable. We then have

$$v_i = \left[ C_{v,i}(sI - A_v - HC_{v,i})^{-1}H + 1 \right]^{-1}\left[ C_{v,i}(sI - A_v - HC_{v,i})^{-1}(B_v + HD_{v,i}) + D_{v,i} \right]$$

Then we define $W_n$ as the transfer matrix with rows $v_{n,i}$ defined by

$$v_{n,i} = C_{v,i}(sI - A_v - HC_{v,i})^{-1}(B_v + HD_{v,i}) + D_{v,i}$$

Then a suitable stable $S$ which makes $SG_{f_2}$ diagonal with non-zero diagonal elements is given by

$$S = W_n G_m^L$$

Note that the reason for carefully removing all other dynamics before constructing $W_n$ is that the dynamic order of $W_n$ can in general be very high because each column has the same state space as $W$ so the dynamic order of $W$ is, in the worst case, equal to the number of rows times the dynamic order of $W$.

4. CONCLUSION

Fourteen different fault detection and identification problems are formulated. The motivation to define and study these problems arises from the desire to weaken the requirements on fault detection and identification instead of seeking exact fault detection and identification as considered often in the literature. Weakening of the requirements could lead to the relaxation of solvability conditions. In order to weaken the requirements, we consider three different notions, (1) almost fault detection and identification, (2) generic fault detection and identification, and (3) classwise fault identification. These notions lead to precise formulation of different fault detection
and identification problems while at the same time inject the essential aspects of engineering applications.

We consider the general class of residual generators which are bounded input bounded output stable non-linear operators. Under this general class of residual generators, we develop the necessary and sufficient solvability conditions for each problem formulated here. In all the problems we studied, our non-linear residual generators have a particular structure of a linear dynamic system followed by a non-linear static mapping. We provide a time domain synthesis procedure based on state space methods to construct appropriate residual generators.

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