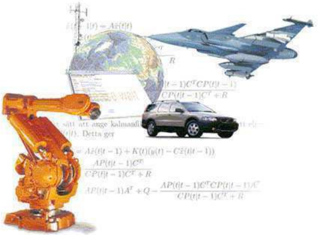
Sensor fusion and parameter inference in nonlinear dynamical systems

– Strategies and concrete examples



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Joint work with (alphabetical order): **Fredrik Gustafsson** (Linköping University), **Michael I. Jordan** (UC Berkeley), **Johan Kihlberg** (Xdin), **Fredrik Lindsten** (Linköping University), **Lennart Ljung** (Linköping University), **Brett Ninness** (University of Newcastle, Australia), **Per-Johan Nordlund** (Saab), **Simon Tegelid** (Xdin) and **Adrian Wills** (MRA, Newcastle, Australia).

Parameter inference

1. The nonlinear Maximum Likelihood (ML) problem
   - Problem formulation
   - Solution using Expectation maximization and a particle smoother
2. The nonlinear Bayesian problem
   - Problem formulation
   - Sketch of solution using MCMC and SMC

Sensor fusion

1. Problem formulation
2. Two industrial application examples

Problem formulation – ML (I/II)

A state space model (SSM) consists of a Markov process $\{x_t\}_{t\geq 1}$ and a measurement process $\{y_t\}_{t\geq 1}$, related according to

$$\begin{aligned} x_{t+1} \mid x_t &\sim f_{\theta,t}(x_{t+1} \mid x_t, u_t), \\ y_t \mid x_t &\sim h_{\theta,t}(y_t \mid x_t, u_t), \\ x_1 &\sim \mu_\theta(x_1), \quad (\theta \sim p(\theta)). \end{aligned}$$

Identification problem: Find θ based on $\{u_{1:T}, y_{1:T}\}$.

ML amounts to solving,

$$\widehat{\theta}^{\text{ML}} = \arg\max_{\theta} \; \log p_\theta(y_{1:T})$$

where the log-likelihood function is given by

$$\log p_\theta(y_{1:T}) = \sum_{t=1}^{T} \log p_\theta(y_t \mid y_{1:t-1})$$

There are at least two challenges with the ML formulation:

1. The one-step prediction PDF $p_\theta(y_t \mid y_{1:t-1})$ has to be computed.
2. In solving the optimization problem

$$\widehat{\theta}^{\text{ML}} = \arg\max_{\theta} \; \log p_\theta(y_{1:T})$$

   the derivatives $\frac{\partial}{\partial \theta} p_\theta(y_t \mid y_{1:t-1})$ are useful.

The **Expectation Maximisation (EM)** algorithm together with a **Particle Smoother (PS)** provides a systematic way of dealing with both of these challenges.

The **Expectation Maximization (EM)** algorithm computes ML estimates of unknown parameters in probabilistic models involving latent variables.

Strategy: Use *structure* inherent in the probabilistic model to separate the original ML problem into *two closely linked subproblems*, each of which is hopefully in some sense more tractable than the original problem.

EM focus on the joint log-likelihood function of the observed variables $y_{1:T}$ and the latent variables $Z \triangleq \{x_1, \ldots, x_T\}$,

$$\ell_\theta(x_{1:T}, y_{1:T}) = \log p_\theta(x_{1:T}, y_{1:T}).$$

Algorithm 1 Expectation Maximization (EM)

1. **Initialise:** Set $i = 1$ and choose an initial θ^1.
2. **While** not converged **do:**
   (a) **Expectation (E) step:** Compute

$$\begin{aligned}\mathcal{Q}(\theta, \theta^i) &= \mathrm{E}_{\theta^i}\left[\log p_\theta(x_{1:T}, y_{1:T}) \mid y_{1:T}\right] \\ &= \int \log p_\theta(x_{1:T}, y_{1:T}) p_{\theta^i}(x_{1:T} \mid y_{1:T}) \mathrm{d}x_{1:T}\end{aligned}$$

   (b) **Maximization (M) step:** Compute

$$\theta^{i+1} = \underset{\theta \in \Theta}{\arg\max}\ \mathcal{Q}(\theta, \theta^i)$$

   (c) $i \leftarrow i + 1$

In computing the $\mathcal{Q}$-function

$$
\begin{aligned}
\mathcal{Q}(\theta, \theta^i) &= \mathrm{E}_{\theta^i}\left[\log p_\theta(x_{1:T}, y_{1:T}) \mid y_{1:T}\right] \\
&= \int \log {\color{blue} p_\theta(x_{1:T}, y_{1:T})} {\color{red} p_{\theta^i}(x_{1:T} \mid y_{1:T})} \mathrm{d}x_{1:T},
\end{aligned}
$$

we start by noting that

$$
\begin{aligned}
{\color{blue}\log p_\theta(x_{1:T}, y_{1:T})} &= \log p_\theta(y_{1:T} \mid x_{1:T}) + \log p_\theta(x_{1:T}) \\
&= \log p_\theta(x_1) + \sum_{t=1}^{T-1} \log p_\theta(x_{t+1} \mid x_t) + \sum_{t=1}^{T} \log p_\theta(y_t \mid x_t)
\end{aligned}
$$

This results in the following expression for the $\mathcal{Q}$-function

$$\mathcal{Q}(\theta, \theta^i) = I_1 + I_2 + I_3,$$

where

$$I_1 = \int \log p_\theta(x_1) p_{\theta^i}(x_1 \mid y_{1:N}) dx_1,$$

$$I_2 = \sum_{t=1}^{T-1} \int \int \log p_\theta(x_{t+1} \mid x_t) p_{\theta^i}(x_{t+1}, x_t \mid y_{1:N}) dx_t dx_{t+1},$$

$$I_3 = \sum_{t=1}^{T} \int \log p_\theta(y_t \mid x_t) p_{\theta^i}(x_t \mid y_{1:N}) dx_t.$$

Nonlinear state smoothing problem, which we approximately solve using sequential Monte Carlo (here, **particle smoothers**).

The particle filter provides an approximation of the filter PDF $p(x_t \mid y_{1:t})$, when the state evolves according to an SSM,

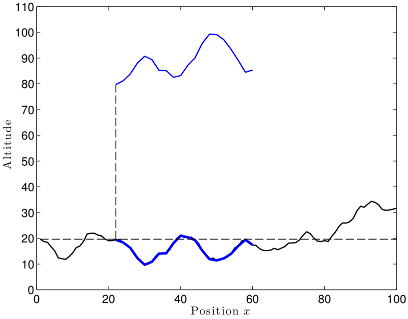
$$\begin{aligned} x_{t+1} \mid x_t &\sim f_{\theta,t}(x_{t+1} \mid x_t, u_t), \\ y_t \mid x_t &\sim h_{\theta,t}(y_t \mid x_t, u_t), \\ x_1 &\sim \mu_\theta(x_1). \end{aligned}$$

The particle filter maintains an empirical distribution made up N samples (particles) and corresponding weights

$$\widehat{p}^N(x_t \mid y_{1:t}) = \sum_{i=1}^{N} w_t^i \delta_{x_t^i}(x_t).$$

"Think of each particle as one simulation of the system state. Only keep the good ones."

Consider a toy 1D localization problem.



Dynamic model:

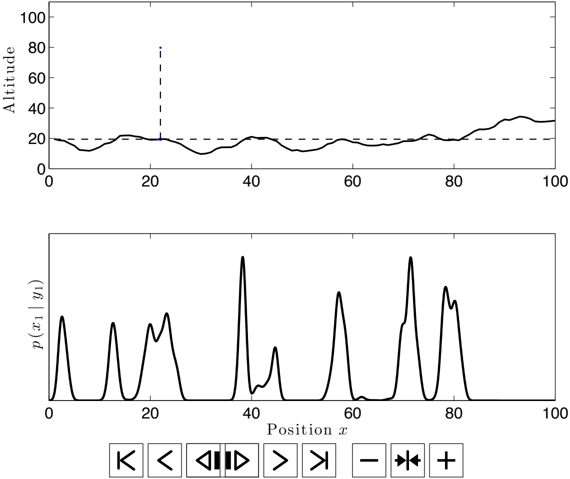
$$x_{t+1} = x_t + u_t + v_t,$$

where x_t denotes position, u_t denotes velocity (known), $v_t \sim \mathcal{N}(0, 5)$ denotes an unknown disturbance.

Measurements:

$$y_t = h(x_t) + e_t.$$

where $h(\cdot)$ denotes the world model (here the terrain height) and $e_t \sim \mathcal{N}(0, 1)$ denotes an unknown disturbance.



Highlights two **key capabilities** of the PF:

1. Automatically handles an unknown and dynamically changing number of hypotheses.
2. Work with nonlinear/non-Gaussian models.

Using particle filters and particle smoothers we straightforwardly obtain the following approximations

$$p_{\theta^i}(x_1 \mid y_{1:T}) \approx \widehat{p}^N_{\theta^i}(x_1 \mid y_{1:T}) = \sum_{i=1}^{N} w^i_{1|T} \delta_{x^i_1}(x_1),$$

$$p_{\theta^i}(x_{t:t+1} \mid y_{1:T}) \approx \widehat{p}^N_{\theta^i}(x_{t:t+1} \mid y_{1:T}) = \sum_{i=1}^{N} w^i_{t|T} \delta_{x^i_{t:t+1}}(x_{t:t+1}).$$

The particle smoother employed is of the forward filtering backward simulation (FFBS) type.

Inserting the above approximations into the integrals yields the approximation we are looking for,

$$
\begin{aligned}
\widehat{I}_1 &= \int \log p_\theta(x_1) \sum_{i=1}^{N} w_{1|T}^i \delta_{x_1^i}(x_1) \mathrm{d}x_1 \\
&= \sum_{i=1}^{N} w_{1|T}^i \log p_\theta(x_1^i), \\
\widehat{I}_3 &= \sum_{t=1}^{T} \int \log p_\theta(y_t \mid x_t) \sum_{i=1}^{N} w_{t|T}^i \delta_{x_t^i}(x_t) \mathrm{d}x_t \\
&= \sum_{t=1}^{T} \sum_{i=1}^{N} w_{t|T}^i \log p_\theta(y_t \mid x_t^i),
\end{aligned}
$$

and similarly for I_2.

Algorithm 2 EM for identifying nonlinear systems

1. **Initialise:** Set $i = 1$ and choose an initial θ^1.
2. **While** not converged **do:**

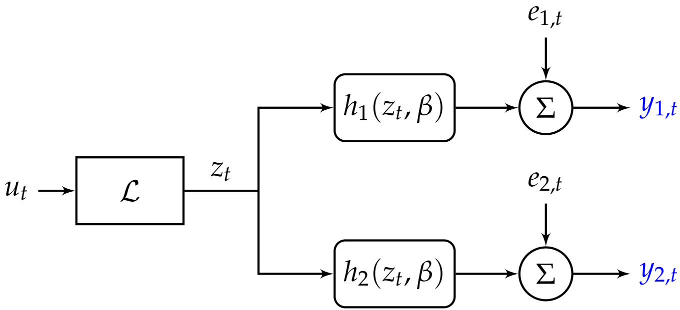
   (a) **Expectation (E) step:** Run a FFBS PS and compute

   $$\widehat{\mathcal{Q}}(\theta, \theta^i) = \widehat{I}_1(\theta, \theta^i) + \widehat{I}_2(\theta, \theta^i) + \widehat{I}_3(\theta, \theta^i)$$

   (b) **Maximization (M) step:** Compute $\theta^{i+1} = \arg\max_{\theta \in \Theta} \widehat{\mathcal{Q}}(\theta, \theta^i)$ using an off-the-shelf numerical optimization algorithm.

   (c) $i \leftarrow i + 1$

Thomas B. Schön, Adrian Wills and Brett Ninness. **System Identification of Nonlinear State-Space Models**. *Automatica*, 47(1):39-49, January 2011.

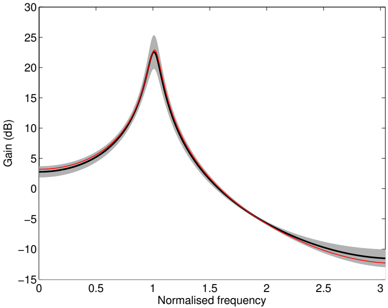


$$
\begin{aligned}
x_{t+1} &= \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} x_t \\ u_t \end{pmatrix}, & & u_t \sim \mathcal{N}(0, Q), \\
z_t &= Cx_t, \qquad y_t = h(z_t, \beta) + e_t, & & e_t \sim \mathcal{N}(0, R).
\end{aligned}
$$

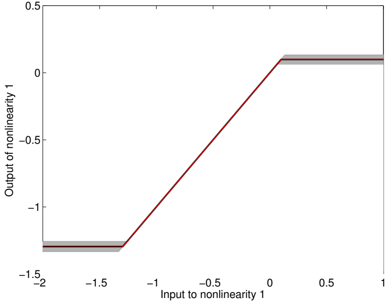
Identification problem: Find A, B, C, β, Q, and R based on $\{y_{1,1:T}, y_{2,1:T}\}$ using EM.

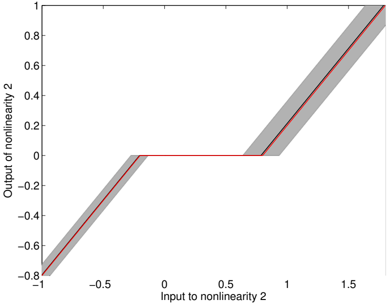
Example – blind Wiener identification (II/III)

- Second order LGSS model with complex poles.
- Employ the EM-PS with $N = 100$ particles.
- EM-PS was terminated after 100 iterations.
- Results obtained using $T = 1000$ samples.
- The plots are based on 100 realizations of data.
- Nonlinearities (dead-zone and saturation) shown on next slide.



Bode plot of estimated mean (black), true system (red) and the result for all 100 realisations (gray).





Estimated mean (black), true static nonlinearity (red) and the result for all 100 realisations (gray).

Adrian Wills, Thomas B. Schön, Lennart Ljung and Brett Ninness. **Identification of Hammerstein-Wiener Models**. *Automatica*, 49(1): 70-81, January 2013.

Parameter inference

1. The nonlinear Maximum Likelihood (ML) problem
   - Problem formulation
   - Solution using Expectation maximization and a particle smoother
2. **The nonlinear Bayesian problem**
   - Problem formulation
   - Sketch of solution using MCMC and SMC

Sensor fusion

1. Problem formulation
2. Two industrial application examples

Bayesian model: θ is a random variable with a prior density $p(\theta)$.

The **goal** in Bayesian modeling is to compute the posterior $p(\underbrace{\theta, x_{1:T}}_{\triangleq \eta} \mid y_{1:T}) = p(\eta \mid y_{1:T})$ (or one of its marginals).

Bayesian modeling/identification amounts to:

1. Find an expression for the likelihood $p(y_{1:T} \mid \eta)$.
2. Assign priors $p(\eta)$ to all unknown stochastic variables η present in the model.
3. Determine the posterior distribution $p(\eta \mid y_{1:T})$.

The **key challenge** is that there is no closed form expression available for the posterior.

Bayesian problem formulation again

Consider a Bayesian SSM

$$\begin{aligned} x_{t+1} \mid x_t &\sim f_{\theta,t}(x_{t+1} \mid x_t, u_t), \\ y_t \mid x_t &\sim h_{\theta,t}(y_t \mid x_t, u_t), \\ x_1 &\sim \mu_\theta(x_1), \\ \theta &\sim p(\theta). \end{aligned}$$

We observe $D_T \triangleq \{u_{1:T}, y_{1:T}\}$.

Goal: Compute the posterior $p(\theta, x_{1:T} \mid D_T)$.

Solution used here – Gibbs sampler

Markov chain Monte Carlo (MCMC) methods allows us to generate samples from an arbitrary target distribution by simulating a Markov chain.

Gibbs sampling (blocked) for SSMs amounts to iterating

- Draw $\theta[m] \sim p(\theta \mid x_{1:T}[m-1], D_T)$,
- Draw $x_{1:T}[m] \sim p(x_{1:T} \mid \theta[m], D_T)$.

The result is a Markov chain

$$\{\theta[m], x_{1:T}[m]\}_{m\geq 1}$$

with $p(\theta, x_{1:T} \mid D_T)$ as its stationary distribution!

Whenever you are working on an algorithm for nonlinear systems, **always** make sure that it solves the simple LGSS problem first!

Consider a fully parameterized LGSS model $(\theta \triangleq \{\Gamma, \Pi\})$.

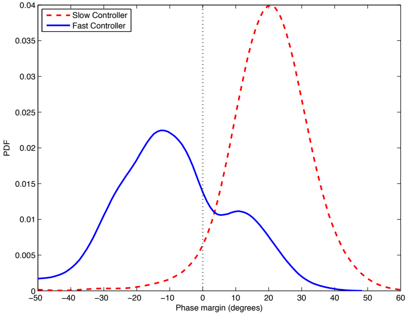
$$\begin{pmatrix} x_{t+1} \\ y_t \end{pmatrix} \mid x_t \sim \mathcal{N}\left(\begin{pmatrix} x_{t+1} \\ y_t \end{pmatrix} \mid \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{\Gamma} \begin{pmatrix} x_t \\ u_t \end{pmatrix}, \underbrace{\begin{pmatrix} Q & S \\ S^{\mathsf{T}} & R \end{pmatrix}}_{\Pi} \right).$$

The posterior distribution $p(\theta \mid D_T)$ is computed using (blocked) Gibbs sampling,

- Draw $\theta[m] \sim p(\theta \mid x_{1:T}[m-1], D_T)$,
- Draw $x_{1:T}[m] \sim p(x_{1:T} \mid \theta[m], D_T)$.

The estimate of the posterior $\widehat{p}(\theta \mid \mathcal{D})$ allows us to access non-standard objects like the PDF of the phase margin $p(\varphi \mid \mathcal{D})$.

- Plots of the phase margin PDFs for two different controllers (one fast and one slow) designed using the identified model $\widehat{p}(\theta \mid \mathcal{D})$.
- Allows us to compute the probability for the closed loop system to go unstable.



Adrian Wills, Thomas B. Schön, Fredrik Lindsten and Brett Ninness, **Estimation of Linear Systems using a Gibbs Sampler**, *Proceedings of the 16th IFAC Symposium on System Identification (SYSID)*, Brussels, Belgium, July 2012.

What would a Gibbs sampler for a general nonlinear/non-Gaussian SSM look like?

- Draw $\theta[m] \sim p(\theta \mid x_{1:T}[m-1], D_T)$,
- Draw $x_{1:T}[m] \sim p(x_{1:T} \mid \theta[m], D_T)$.

Problem: $p(x_{1:T} \mid \theta, D_T)$ is not available!!

Idea: Approximate $p(x_{1:T} \mid \theta, D_T)$ using a particle smoother (PS).

(Non-trivial) solution: Careful and clever analysis of how to combine MCMC and PF/PS results in the PMCMC family of algorithms.

Facts about Particle Markov Chain Monte Carlo (PMCMC) samplers:

- Provides a systematic and provably correct combination of PF/PS and MCMC.
- Standard MCMC samplers on non-standard spaces.
- Constitutes a family of Bayesian inference methods, including
    - Particle Independent Metropolis Hastings (PIMH)
    - Particle Marginal Metropolis Hastings (PMMH)
    - **Particle Gibbs (PG)**

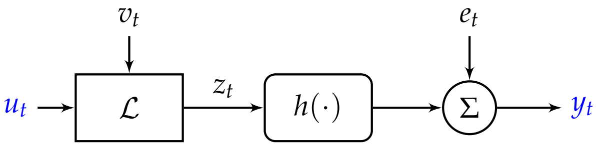
Christophe Andrieu, Arnaud Doucet and Roman Holenstein, **Particle Markov chain Monte Carlo methods**, *Journal of the Royal Statistical Society: Series B*, 72:269-342, 2010.

PG-BS sampler targeting $p(\theta, x_{1:T} \mid D_T)$.

- Conditional particle filter and backward simulation
    - Run a conditional PF, targeting $p(x_{1:T} \mid \theta, D_T)$;
    - Run a backward simulator to sample $x^\star_{1:T}$;
- Draw $\theta^\star \sim p(\theta \mid x^\star_{1:T}, D_T)$.

Powerful and important property of PG-BS: Provably convergent for any $N \geq 2$ particles and it works in practice!

Fredrik Lindsten and Thomas B. Schn. **On the use of backward simulation in the particle Gibbs sampler**. *Proceedings of the 37th International Conference on Acoustics, Speech, and Signal Processing (ICASSP)*, Kyoto, Japan, March 2012.



Parametric LGSS and a nonparametric static nonlinearity:

$$
\begin{aligned}
x_{t+1} &= \underbrace{\begin{pmatrix} A & B \end{pmatrix}}_{\Gamma} \begin{pmatrix} x_t \\ u_t \end{pmatrix} + v_t, &\qquad v_t &\sim \mathcal{N}(0, Q), \\
z_t &= C x_t. \\
y_t &= h(z_t) + e_t, &\qquad e_t &\sim \mathcal{N}(0, R).
\end{aligned}
$$

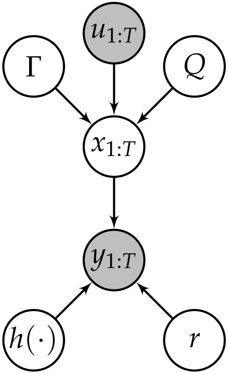
First step towards a fully data driven model (the order of the LGSS model is still assumed known).

Parameters: $\theta = \{\Gamma, Q, r, h(\cdot)\}$.

Bayesian model specified by priors:

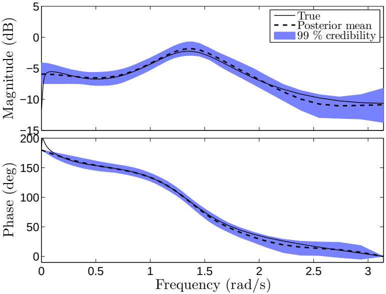
- Conjugate priors for $\Gamma = [A\ B]$, Q and r,
  - $p(\Gamma, Q) =$ Matrix-normal inverse-Wishart
  - $p(r) =$ inverse-Wishart
- Gaussian process prior on $h(\cdot)$,

$$h(\cdot) \sim \mathcal{GP}(z, k(z, z')).$$

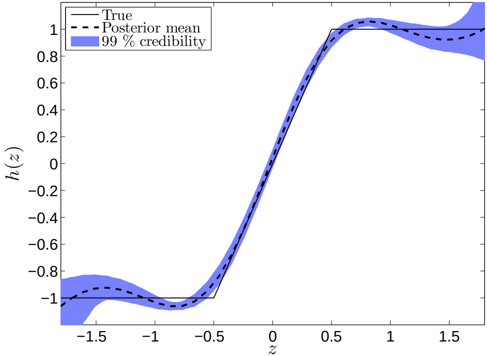


Example – known model order (I/II)

- Bayesian semiparametric model with conjugate prior (MNIW).
- 6^{th} order LGSS model and a saturation.
- Using $T = 1000$ measurements.
- Employ the PG-BS sampler with $N = 15$ particles.
- Run 15000 MCMC iterations, discard 5000 as burn-in.



True Bode diagram of the linear system (solid black), estimated mean (dashed black) and 99% credibility interval (blue).

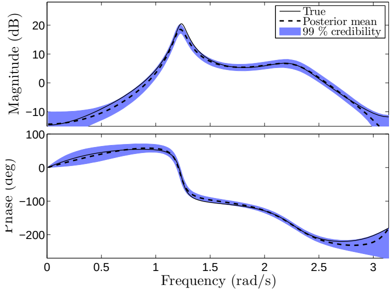


True static nonlinearity (solid black), estimated posterior mean (dashed black) and 99% credibility interval (blue).

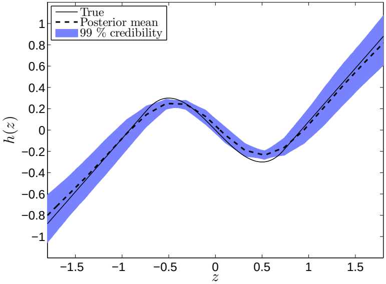
Fredrik Lindsten, Thomas B. Schön and Michael I. Jordan. **A semiparametric Bayesian approach to Wiener system identification**. *Proceedings of the 16th IFAC Symposium on System Identification (SYSID)*, Brussels, Belgium, July 2012.

Semiparametric model 2 - unknown model order

Show movie



Bode diagram of the 4th-order linear system. Estimated mean (dashed black), true (solid black) and 99% credibility intervals (blue).



Static nonlinearity (non-monotonic), estimated mean (dashed black), true (black) and the 99% credibility intervals (blue).

Fredrik Lindsten, Thomas B. Schön and Michael I. Jordan. **Bayesian semiparametric Wiener system identification**. *Automatica*, 2012 (in revision).

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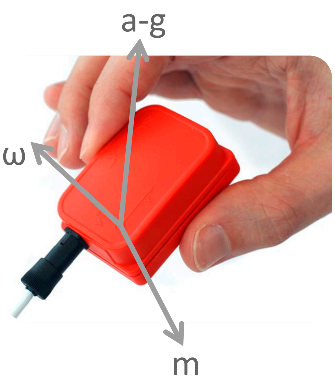
Sensor fusion

1. Problem formulation
2. Two industrial application examples

Aim: Motion capture, find the motion (position, orientation, velocity and acceleration) of a person (or object) over time.

Industrial partner: Xsens Technologies.



Illustrative example (II/II)

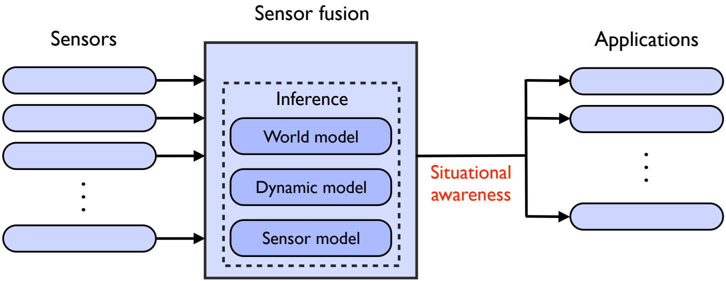
1. Only making use of the inertial information.
2. Inertial + biomechanical model.
3. Inertial + biomechanical model + world model.
4. Inertial + biomechanical model + world model + UWB.

Illustrates the importance of suitable models.

Sensor fusion – definition

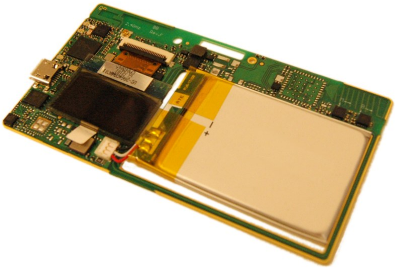
Definition (Sensor fusion)

Sensor fusion is the process of using information from **several different** sensors to **infer** what is happening (this typically includes finding states of dynamical systems and various static parameters).

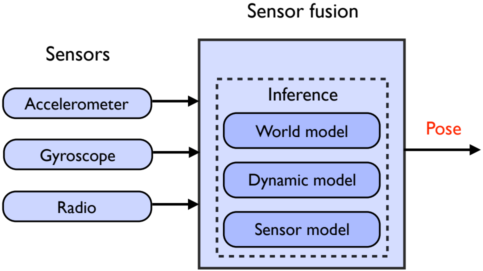


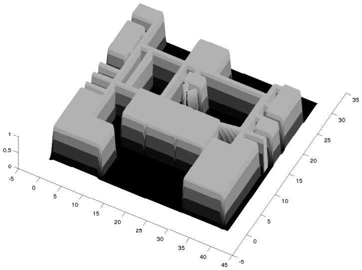
Aim: Compute the position of a person moving around indoors using sensors (inertial, magnetometer and radio) located in an ID badge and a map.

Industrial partner: Xdin



Example – Indoor localization (II/II)





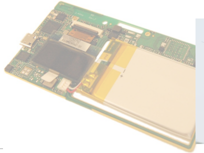
Show movie

Conclusions

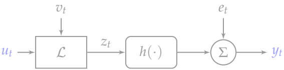
- Maximum likelihood identification:
  - EM (nonlinear optimization and PS)
- Bayesian identification:
  - PMCMC = combination of MCMC and PF/PS
- Solved various Wiener identification problems for illustration.
- Sensor fusion
  - Model the dynamics, the sensors and the world. Solve the resulting inference problem.
  - The industrial utility of this technology is growing as we speak!
- Much interesting research **remains to be done**!!

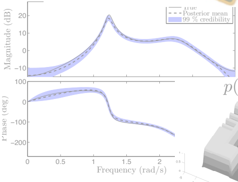
In this talk I introduced strategies and showed a few concrete examples. Should you be interested in the details, I have a PhD course on this topic, `users.isy.liu.se/rt/schon/course_CIDS.html`

Thank you for your attention!



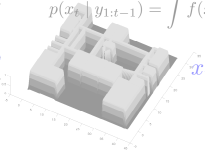
$$\mathcal{Q}(\theta,\theta^i) = \mathrm{E}_{\theta^i}\left[\log p_\theta(x_{1:T},y_{1:T}) \mid y_{1:T}\right] = \int \log p_\theta(x_{1:T},y_{1:T}) p_{\theta^i}(x_{1:T} \mid y_{1:T}) \mathrm{d}x_{1:T},$$





$$p(x_t \mid y_{1:t}) = \frac{h(y_t \mid x_t)p(x_t \mid y_{1:t-1})}{p(y_t \mid y_{1:t-1})},$$

$$p(x_t \mid y_{1:t-1}) = \int f(x_t \mid x_{t-1})p(x_{t-1} \mid y_{1:t-1})\mathrm{d}x_{t-1},$$



$$x_{t+1} \mid x_t \sim f_\theta(x_{t+1} \mid x_t, u_t),$$
$$y_t \mid x_t \sim h_\theta(y_t \mid x_t, u_t).$$