Explicit Wei-Norman formulæ for matrix Lie groups via Putzer's method

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Abstract

The Wei-Norman formula locally relates the Magnus solution of a system of linear time-varying ODEs with the solution expressed in terms of products of exponentials by means of a set of nonlinear differential equations in the parameters of the two types of solutions. A closed form expression of such formula is proposed based on the use of Putzer's method.

Key words: matrix Lie groups, time-varying differential equations, Magnus expansions, Wei-Norman formula.

1 Introduction

The use of Wei-Norman formulæ [28] is ubiquitous in control and systems theory, see [6,4,13,17,9] for a few examples. Essentially, they are of importance whenever one wants to establish local coordinates on a smooth manifold on which a finite dimensional group of transformations is acting (the most trivial example being a matrix transition Lie group in linear time-varying differential equations). In fact, a local chart on the manifold is induced by the so-called canonical coordinates of the second kind via the exponential map of the Lie group and the Lie group action. In case the manifold is itself a group, then canonical coordinates of the second kind and, more generally, "well posed" products of exponentials form the basis for the most common way of parameterizing the Lie group itself. For example for the group of rigid body motions in 3 dimensions, they produce the various sets of Euler angles. In control theory, a few uses of Wei-Norman formulæ are the following. In the computation of reachable sets of control systems, integral versions of the Wei-Norman formulæ constitute

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the mathematically sound way to pass from the flow corresponding to continuous inputs to the so-called polysystem (via the approximation lemma) driven by concatenations of piecewise constant controls. In the analysis of rigid robotic chains, the products of exponentials formula of Brockett [5] is the basis for the geometric modeling developed in the book [20] (the Jacobian of a robotic chain which is extensively used for the differential kinematics is basically a Wei-Norman formula). Also in underactuated motion planning they have been used, for example in [17]. Aside from control theory, the Wei-Norman formula appears in several different contexts in the literature, sometimes quoted just as an application of the Campbell-Baker-Hausdorff formula, see [7,10,11,23] just to mention a few. Recently, they have also attracted the interest of the numerical algebra community [14,21] as a way to provide numerical algorithms that respect the geometric structure of a manifold or of a group manifold.

In essence, the Wei-Norman formula provides an explicit relationship between the so-called Magnus expansion, i.e. local solutions of a linear time-varying system of ODEs expressed by means of a single exponential, and a "complete" product of exponentials expansion, i.e. involving a complete basis of the corresponding finite dimensional Lie algebra. Such relation is given by a set of nonlinear differential equations in the parameters of the two expansions and essentially represents the Jacobian of the change of coordinates from single exponential to product of exponentials representations. The basic ingredients needed in the formula are rather elementary: an infinitesimal matrix representation of a Lie group and a closed form expression of a matrix exponential. In fact, for low dimensional Lie groups, hand computations of the Wei-Norman formulæ have been carried out frequently and extensively used, especially when trying to write differential equations in terms of group parameterizations, like Euler angles for orthogonal groups. Also more systematic treatements are available in the literature, see [6,13,24,3], as well as numerically sound methods [21] and numerical approximations [8,29]. which avoid exact computations of exponentials while choosing "structure preserving" approximations.

In this study, we propose an alternative method to compute such formula explicitly and systematically, requiring only on the structure constants of the Lie algebra and not involving any infinite sum. The *rationale* behind our method is a technique to compute in closed form one parameter groups of automorphisms, i.e. exponentials of the matrices of the adjoint representation of any linear Lie algebra, based on Putzer's method [22,25], suitably modified in order to deal with multiple eigenvalues. The paper is organized as follows: in Section 2 the Wei-Norman formula is recalled for invariant differential equations on matrix Lie groups; in Section 3 the computation of the exponentials of adjoint operators in terms of structure constants is carried out and it is used in Section 4 to express the Wei-Norman formula. Finally, in Section 5, the example of the Special Euclidean group of rigid body transformations in 3 dimensions is treated.

2 Wei-Norman formulæ

Given a system of invariant parametric ordinary differential equations evolving on an n-dimensional matrix Lie group G

$$\begin{aligned} \dot{g}(t) &= \sum_{i=1}^{m} A_i u_i(t) g(t) \; , \qquad g(t) \in G \\ g(0) &= g_0 \end{aligned} \tag{1}$$

where $A_1, \dots A_m$ are part of an n-dimensional matrix basis $\{A_1, \dots, A_n\}$ of the Lie algebra $\mathfrak{g}$ of G and the $u_i(t)$ are real valued parameters, there are two types of local representations of the solution of (1) which are of particular importance. They rely respectively on the single exponential representation (due to Magnus [18]) and on the product of exponentials formulation (due to Wei-Norman [28]) If $m = n$, they also correspond to the so-called canonical coordinates of the first and second kind. In fact, for matrix groups the exponential map coincides with the ordinary matrix exponential and locally, in a neighborhood $\mathcal{N}$ of the identity, it provides a diffeomorphism so that we can define the *coordinate mapping* for $g(t) \in \mathcal{N}$

$$g(t) = e^{\psi_1(t) A_1 + \psi_2(t) A_2 + \ldots + \psi_n(t) A_n} \tag{2}$$

where $\psi_i, \; i = 1, \dots, n$ are called *local coordinates of the first kind* for G around the identity, relative to the basis $A_1, \dots A_n$. If instead we write:

$$g(t) = e^{\theta_1(t) A_1} e^{\theta_2(t) A_2} \dots e^{\theta_n(t) A_n} \; , \qquad g(t) \in \mathcal{N} \tag{3}$$

then $\theta_i, \; i = 1, \dots, n$ are called *canonical coordinates of the second kind* on G. In both cases, right translation can be used to construct an entire atlas for all G.

The Wei-Norman formula [28] reformulates the differential equation (1) in terms of local coordinates of the second kind and it is obtained from the following proposition:

Lemma 1 (Wei-Norman) *Let $g(t) \in G$ be the solution of the system (1) starting with initial condition $g(0) = g_0$. Then there exists a neighborhood of $t = 0$ in which $g(t)$ can be expressed as a product of exponentials*

$$g(t) = e^{\gamma_1(t) A_1} e^{\gamma_2(t) A_2} \dots e^{\gamma_n(t) A_n} g_0 \tag{4}$$

The Wei-Norman coordinate functions $\gamma_i(t)$, $i = 1, \dots, n$, are scalar functions of t and evolve according to the set of differential equations on $\mathbb{R}^n$:

$$\begin{bmatrix} \dot{\gamma}_1(t) \\ \vdots \\ \dot{\gamma}_n(t) \end{bmatrix} = \Xi(\gamma_1(t), \dots, \gamma_n(t))^{-1} \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} \tag{5}$$

where $\gamma_i(0) = 0$ and the matrix $\Xi(\cdot)$ is a real analytic function of the γ_i.

The calculation of $\Xi(\gamma(t))$ can be done in the following way (see [6]): compare the expression

$$\dot{g} = \sum_{i=1}^{n} A_i u_i(t)\, g(t) \tag{6}$$

obtained from (1) by adding null parameters $u_{m+1} = \ldots = u_n = 0$, with the derivative of g with respect to the product of exponentials (4):

$$\begin{aligned}\dot{g}(t) = \Big(&\dot{\gamma}_1(t) A_1 e^{\gamma_1(t)A_1} e^{\gamma_2(t)A_2} \ldots e^{\gamma_n(t)A_n} + \dot{\gamma}_2(t) e^{\gamma_1(t)A_1} A_2 e^{\gamma_2(t)A_2} \ldots e^{\gamma_n(t)A_n} + \\ &+ \ldots + \dot{\gamma}_n(t) e^{\gamma_1(t)A_1} \ldots e^{\gamma_{n-1}(t)A_{n-1}} A_n e^{\gamma_n(t)A_n}\Big)\, g_0.\end{aligned}$$

In the following we will drop the time dependence in the γ_i and u_i. Inserting the identity terms $e^{\gamma_i A_i} e^{-\gamma_i A_i}$ where needed, and using the adjoint map $\mathrm{Ad}_g \; : \mathfrak{g} \to \mathfrak{g}$, $A \mapsto \mathrm{Ad}_g A = gAg^{-1}$, we can extract $g(t)$ on the right

$$\dot{g}(t) = \Big(A_1 \dot{\gamma}_1 + \mathrm{Ad}_{e^{\gamma_1 A_1}} A_2 \dot{\gamma}_2 + \mathrm{Ad}_{e^{\gamma_1 A_1} e^{\gamma_2 A_2}} A_3 \dot{\gamma}_3 + \ldots + \mathrm{Ad}_{\prod_{i=1}^{n-1} e^{\gamma_i A_i}} A_n \dot{\gamma}_n \Big)\, g(t). \tag{7}$$

Next, we need to compare (6) and (7) along each of the basis elements of $\mathfrak{g}$, i.e., we need to compute the contribution of the adjoint operators

$$\mathrm{Ad}_{\left(\prod_{i=j}^{n} e^{\gamma_i A_i}\right)} A_j = \prod_{i=j}^{n} \left(e^{\gamma_i \mathrm{ad}_{A_i}} \right) A_j \tag{8}$$

in terms of the A_i using the formula

$$e^{\gamma_i \mathrm{ad}_{A_i}} A_j = e^{A_i \gamma_i} A_j e^{-A_i \gamma_i} = \sum_{l=0}^{\infty} \frac{\gamma_i^l}{l!} \mathrm{ad}_{A_i}^l A_j = \sum_{l=0}^{\infty} \frac{\gamma_i^l}{l!} c_{i\,j}^{k_1} c_{i\,k_1}^{k_2} \ldots c_{i\,k_{l-1}}^{k_l} A_{k_l} \tag{9}$$

where $\mathrm{ad}_{A_i}^0 A_j = A_j$, $\mathrm{ad}_{A_i}^1 A_j = [A_i, A_j] = c_{ij}^k A_k$ and $\mathrm{ad}_{A_i}^l A_j = \left[A_i, ad_{A_i}^{l-1} A_j\right] = c_{i\,j}^{k_1} c_{i\,k_1}^{k_2} \ldots c_{i\,k_{l-1}}^{k_l} A_{k_l}$, with the summation convention enforced over repeated indexes. $c_{ij}^k = -c_{ji}^k$ are the structure constants of $\mathfrak{g}$ associated with the basis $A_1, \ldots, A_n$. For adjoint maps of exponentials, in (9) we have used the notation $\mathrm{Ad}_{e^{\gamma_i A_i}} = e^{\gamma_i \mathrm{ad}_{A_i}}$. Iterating this procedure, it is possible to write (8) in terms of infinite series of the structure constants of the Lie algebra and therefore to obtain $\Xi(\gamma(t))$, at least in principle. It is also easy to realize that explicit expressions for the coefficients of the A_k in (9) are in general difficult to calculate with this method, because of the infinite sums involved.

Notice the compatibility of the initial conditions in the two expressions (6) and (7) for $\dot{g}(t)$, which implies that $\Xi(\gamma(0)) = I$ and therefore $\Xi(\cdot)$ locally invertible. Notice moreover that, by the right invariance, the initial state g_0 of the system does not appear in $\Xi(\gamma(t))$. If $\mathfrak{g}$ is solvable, then there exist coordinate functions γ_i that are globally valid, while this is not true for semisimple Lie algebras. In this case, the nonsingularity of Ξ has to be checked at the point of application.

$\Xi(\cdot)$ changes according to the order chosen for the basis elements, and therefore also the combination of u_i that solves (5) looks different with different basis ordering. Only one of the possible admissible combinations of the u_i is captured at a time by the method just presented.

3 Computation of the exponentials in the adjoint representation

The adjoint representation of a linear Lie algebra $\mathfrak{g}$ is a derivation of the algebra and as such it is the infinitesimal generator of a one-parameter group of automorphisms. In the basis $A_1, \ldots, A_n$ of $\mathfrak{g}$, the corresponding basis of the adjoint representation is given by the matrices ad_{A_i}. If γ_i is the real valued parameter associated with ad_{A_i}, then the one-parameter group of automorphisms is described by the matrix exponential $e^{\mathrm{ad}_{A_i}}$. Scope of this section is to provide an explicit formula for the $e^{\mathrm{ad}_{A_i}}$, not involving any infinite summation.

There exist many techniques to compute the exponential of a square matrix, see the classical survey [19] and the recent "classroom notes" of SIAM Review [12,16] and references therein. The simplest method is perhaps the use of the Sylvester formula [2], p.233, which requires only the knowledge of the eigenvalues of the matrix. Alternatively, one can for example use the spectral decomposition of normal operators i.e. compute the Jordan form by similarity transformations, but this requires the knowledge of eigenvalues and eigenvectors. The method we use here relies on the Cayley-Hamilton theorem and consists in expressing the series expansion of $e^{\mathrm{ad}_{A_i}}$ in terms of the first $n-1$ powers of ad_{A_i} with suitable coefficients depending on the coefficients of the characteristic polynomial of ad_{A_i} and on γ_i. This method suits well for the adjoint representation, as the powers of ad_{A_i} are immediately expressed in terms of the structure constants of the Lie algebra. In fact, if c_{ij}^k are the structure constants, the matrices corresponding to the basis elements A_i are $\mathrm{ad}_{A_i} = M_i$ of elements $(M_i)_{kj} = c_{ij}^k$ and the l-th power of ad_{A_i} is given by

$$M_i^l = \left(M_i^l\right)_{kj} = c_{i\,\mu_1}^{k} c_{i\,\mu_2}^{\mu_1} \ldots c_{i\,\mu_{l-1}}^{\mu_{l-2}} c_{i\,j}^{\mu_{l-1}} \tag{10}$$

3.1 Linear representations of commutator operators

Any matrix $B \in \mathfrak{g}$ can be written as $B = b^\mu A_\mu$. If we identify B with the n-dimensional coordinate vector $B \simeq b = \begin{bmatrix} b^1 & \ldots & b^n \end{bmatrix}^T$, then $[A_i,\, B] \simeq \mathrm{ad}_{A_i} b = (M_i)b$, i.e. the Lie bracket gives another column vector $(M_i)_{k\mu} b^\mu = c_{i\mu}^k b^\mu = \begin{bmatrix} c_{i\mu}^1 b^\mu & c_{i\mu}^2 b^\mu & \ldots & c_{i\mu}^n b^\mu \end{bmatrix}^T$, while the Lie bracket of $D \simeq d = \begin{bmatrix} d^1 & \ldots & d^n \end{bmatrix}^T$ with B looks like:

$$[D,\, B] \simeq \mathrm{ad}_D b = d^\nu (M_\nu) b = \begin{bmatrix} d^\nu c_{\nu\mu}^1 b^\mu & d^\nu c_{\nu\mu}^2 b^\mu & \ldots & d^\nu c_{\nu\mu}^n b^\mu \end{bmatrix}^T$$

As an example, compute $[B,\, A_i] = -[A_i,\, B]$. Since $A_i \simeq \mathrm{e}_i$, the standard basis vector of $\mathbb{R}^n$, we have:

$$[B,\, A_i] \simeq b^\nu (M_\nu)_{ki} = b^\nu \begin{bmatrix} c^1_{\nu i} & \dots & c^n_{\nu i} \end{bmatrix}^T = - \begin{bmatrix} c^1_{i\nu} & \dots & c^n_{i\nu} \end{bmatrix}^T b^\nu = -(M_i)_{k\nu} b^\nu \simeq -[A_i,\, B] \quad (11)$$

This way of representing Lie brackets via linear operators on vectors of coordinates is well-known. In the physics literature these linear operators are sometimes referred to as commutator superoperators. If the dimension of the Lie algebra $\mathfrak{g}$ is n, then the typical size of the corresponding superoperators is $n \times n$. Since we are concerned only with real Lie algebras, the superoperators are always matrices of real entries.

3.2 Exponential of the ad_A *matrix*

The adjoint representation corresponds to a derivation of the Lie algebra, given $A \in \mathfrak{g}$,

$$\mathrm{ad}_A = \frac{d}{d\gamma} \left(\mathrm{Ad}_{e^{\gamma A}}\right)\Big|_{\gamma=0}, \qquad \gamma \in \mathbb{R},$$

and $\mathrm{ad}_A = M$ is the infinitesimal generator of the corresponding one-parameter group of automorphisms

$$\mathrm{Ad}_{e^{A\gamma}} = e^{\gamma \mathrm{ad}_A} = e^{\gamma M} = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} \mathrm{ad}_A^k = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} M^k. \quad (12)$$

From (8), the computation of Ξ requires the explicit evaluation of the exponentials (12). The expansion (12) involves infinite series of elementary commutators. To obtain an explicit closed form expression for it, we write the exponential of a $n \times n$ matrix in terms of its first $n-1$ powers. If the characteristic polynomial is $p(s) = \det(sI - M) = s^n - a_{n-1}s^{n-1} - \ldots - a_1 s - a_0$, with coefficients $a_{n-1} = \mathrm{tr}(M), \ldots, a_0 = (-1)^n \det M$, since M satisfies its own characteristic equation $p(M) = 0$, i.e.,

$$M^n = a_0 I + a_1 M + a_2 M^2 + \ldots + a_{n-1} M^{n-1}, \quad (13)$$

then (12) can always be written as

$$e^{\gamma M} = \sum_{k=0}^{n-1} \beta_k M^k \quad (14)$$

for suitable $\beta_k = \beta_k(a_0,\, \ldots, a_{n-1},\, \gamma)$. To obtain the β_k, we rely essentially on Putzer's method [22,25] for closed-forms of exponential matrices. The algorithm is a modification of the one described in [25] in order to incorporate easily matrices M having multiple eigenvalues.

As functions of γ, the β_k are the principal solutions of the differential equation

$$p(D) e^{\gamma M} = 0 \quad (15)$$

where $D = \frac{d}{d\gamma}$ is the differential operator. In fact, since $e^{\gamma M}$ is the solution of the initial value problem

$$\begin{cases} DX & = MX \\ X(0) & = I \end{cases}$$

from $p(M) = 0$ and $D^k e^{\gamma M} = M^k e^{\gamma M}$ equation (15) follows. Furthermore, from (14), evaluating the derivatives at $t = 0$, the initial conditions of the β_j are $\beta_j^{(i-1)}(0) = \delta_j^i$, thus β_j are principal solutions.

The explicit values of the β_k are obtained by the following recursive scheme:

$$\begin{aligned} M^{n+1} &= a_0 M + a_1 M^2 + \ldots + a_{n-1} M^n = \alpha_{n+1,0} I + \alpha_{n+1,1} M + \ldots + \alpha_{n+1,n-1} M^{n-1} \\ M^{n+2} &= \alpha_{n+2,0} I + \alpha_{n+2,1} M + \ldots + \alpha_{n+2,n-1} M^{n-1} \\ &\vdots \\ M^k &= \alpha_{k,0} I + \alpha_{k,1} M + \ldots + \alpha_{k,n-1} M^{n-1} \end{aligned}$$

Stacking together the $\alpha_{k,j}$, $j = 0, \ldots n-1$, we obtain a linear vector difference equation with update matrix C in companion form:

$$\alpha(k+1) = \begin{bmatrix} \alpha_{k+1,0} \\ \alpha_{k+1,1} \\ \vdots \\ \alpha_{k+1,n-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots & 0 & a_0 \\ 1 & 0 & & & a_1 \\ & 1 & & & \\ & & \ddots & & \vdots \\ & & & 1 & a_{n-1} \end{bmatrix} \begin{bmatrix} \alpha_{k,0} \\ \alpha_{k,1} \\ \vdots \\ \alpha_{k,n-1} \end{bmatrix} = C\alpha(k) \tag{16}$$

with initial condition $\begin{bmatrix} \alpha_{0,0} & \alpha_{0,1} & \ldots & \alpha_{0,n-1} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T$. Thus

$$\beta(\gamma) = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} \begin{bmatrix} \alpha_{k,0} \\ \alpha_{k,1} \\ \vdots \\ \alpha_{k,n-1} \end{bmatrix} = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} \alpha(k) = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} C^k \alpha(0) = e^{\gamma C} \alpha(0) \tag{17}$$

Since C is in companion form, the exponential $e^{\gamma C}$ can be evaluated exactly by passing to the Jordan form via Vandermonde matrices. In case the eigenvalues $\lambda_1, \ldots, \lambda_n$ of M are all distinct, then the Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix}$$

has columns that are the eigenvectors of C and diagonalizes C:

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = V^{-1} C^T V \tag{18}$$

Thus

$$e^{\gamma C} = V^{-T} \begin{bmatrix} e^{\gamma\lambda_1} & 0 & \dots & 0 \\ 0 & e^{\gamma\lambda_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{\gamma\lambda_n} \end{bmatrix} V^T$$

Explicit expressions for $V^{-1} = (v_{ij})$ are available in the literature, see [26,27]:

$$v_{ij} = \sum_{k=j}^{n} \frac{a_k \lambda_i^{k-j}}{\prod_{h=1,\, h\neq j}^{n} (\lambda_j - \lambda_h)}$$

The method relies on the fact that the update matrix C of the recursive difference equation (16) has the same characteristic polynomial as M but it is in companion form. This, together with the knowledge of the eigenvalues of M allows to achieve the desired closed form. Notice that the projectors of M, $E_1, \dots, E_n$, given by $E_j = \prod_{k=1,\, k\neq j}^{n} \frac{M-\lambda_k I}{\lambda_j - \lambda_k}$, can also be computed from V^{-T}: $E_j = \sum_{k=1}^{n} v_{jk} M^{k-1}$. From the spectral theorem, projectors are obviously a "preferred" set of n matrices (other than $I, M, \dots, M^{n-1}$) because for them the exponentials $e^{\gamma\lambda_j}$ constitute a fundamental set of solutions of (15):

$$e^{\gamma C} = e^{\gamma\lambda_1} E_1 + \dots + e^{\gamma\lambda_n} E_n. \tag{19}$$

Following [12], in the case of multiple eigenvalues, (18) and (19) cannot be used since V is singular, so that Putzer's method in its standard form [25] does not apply. Instead of V, the so-called confluent Vandermonde matrix W can be used, see [27,15] and references therein. Assume that the characteristic roots $\lambda_1, \dots, \lambda_r$, $r \leq n$ have multiplicities $m_1, \dots m_r$. Then

W consists of r blocks of dimensions $m_\rho \times m_\rho$, $\rho = 1, \ldots r$, and of the form

$$\begin{bmatrix} 1 & 0 & 0 & \ldots 0 \\ \lambda_\rho & 1 & 0 & \\ \lambda_\rho^2 & 2\lambda_\rho & 1 & \\ \lambda_\rho^3 & 3\lambda_\rho^2 & 3\lambda_\rho & \ddots \\ \vdots & & & \\ \lambda_\rho^{n-1} & (n-1)\lambda_\rho^{n-2} & \binom{n-1}{2}\lambda_\rho^{n-3} & 1 \end{bmatrix}$$

where the generic element of position (μ, ν) is given by $\binom{\mu-1}{\nu-1}\lambda_\rho^{\mu-\nu}$. Similarly to (18),

$$W^{-1}F^T W = J$$

where J is the Jordan canonical form $J = \mathrm{diag}\,(J_1, \ldots, J_r)$ of diagonal blocks

$$J_\rho = \begin{bmatrix} \lambda_\rho & 1 & & \\ & \lambda_\rho & \ldots & \\ & & \ddots & 1 \\ & & & \lambda_\rho \end{bmatrix}$$

In this case, the computation of β becomes more involved. Still it can be carried out explicitly.

Notice that quite explicit results can be obtained alternatively by applying a Laplace transform method instead of the scheme (16)-(17), see [1,19].

3.3 Exponentials of the basis elements ad_{A_i}

For the basis elements $A_i \in \mathfrak{g}$, the powers of $M_i = \mathrm{ad}_{A_i}$ are computed in terms of the structure constants in (10). Thus using the notation $\beta_0^{[i]}, \beta_1^{[i]}, \ldots \beta_{n-1}^{[i]}$ for the coefficients obtained in (17) (the upper index is between brackets to indicate that summation is not to be performed in the formulæ below) we have

$$\begin{aligned} e^{\gamma_i \mathrm{ad}_{A_i}} = \left(e^{\gamma_i M_i}\right)_{kj} &= \beta_0^{[i]}\delta_j^k + \beta_1^{[i]} c_{i\,j}^{k} + \beta_2^{[i]} c_{i\,l_1}^{k} c_{i\,j}^{l_1} + \ldots + \beta_{n-1}^{[i]} c_{i\,l_1}^{k} c_{i\,l_2}^{l_1} \ldots c_{i\,j}^{l_{n-2}} \\ &= \sum_{r=0}^{n-1} \beta_r^{[i]} c_{i\,l_1}^{k} c_{i\,l_2}^{l_1} \ldots c_{i\,j}^{l_{r-1}} \end{aligned} \tag{20}$$

where it is intended that $c_{i\,l_1}^{k} c_{i\,l_2}^{l_1} \ldots c_{i\,j}^{l_{r-1}} = \delta_j^k$ for $r = 0$ and $c_{i\,l_1}^{k} c_{i\,l_2}^{l_1} \ldots c_{i\,j}^{l_{r-1}} = c_{ij}^k$ for $r = 1$ (the lower index in $\beta_k^{[i]}$ gives the number of times the structure constants $c_{i\,*}^{*}$ appear in the corresponding term). Also the product of matrices $e^{\gamma_i \mathrm{ad}_{A_i}}$ and $e^{\gamma_l \mathrm{ad}_{A_l}}$ can be expressed in terms of the $\beta_0^{[i]}, \ldots, \beta_{n-1}^{[i]}, \beta_0^{[l]}, \ldots, \beta_{n-1}^{[l]}$ and of the structure constants as follows:

$$e^{\gamma_i \mathrm{ad}_{A_i}} e^{\gamma_l \mathrm{ad}_{A_l}} = \left(e^{\gamma_i M_i} e^{\gamma_l M_l} \right)_{kj} = \sum_{r,\, s=0}^{n-1} \beta_r^{[i]} \beta_s^{[l]} c_{i\, p_1}^{k} c_{i\, p_2}^{p_1} \ldots c_{i\, q}^{p_{r-1}} c_{l\, p_1}^{q} c_{l\, p_2}^{p_1} \ldots c_{l\, j}^{p_{s-1}} \tag{21}$$

with a similar convention as above for $r, s = 0, 1$. Notice how also the "concatenation" of c_{i*}^{*} and c_{l*}^{*} respects the summation convention (indeed (21) represents an ordinary product of square matrices).

4 Explicit expression of the Wei-Norman formula

From Section 2 and with the notation introduced in Section 3, the matrix Ξ of (5) is given by

$$\Xi = \begin{bmatrix} e_1 & e^{\gamma_1 \mathrm{ad}_{A_1}} e_2 & \ldots & \prod_{i=1}^{n-1} e^{\gamma_i \mathrm{ad}_{A_i}} e_n \end{bmatrix}$$

With the closed form expression (20) for $e^{\gamma_i \mathrm{ad}_{A_i}}$, it is possible to compute Ξ explicitly, without any infinite summation. From (7) and (21) it is clear what we have to do: from the matrix products of the $e^{\gamma_i \mathrm{ad}_{A_i}}$, compute all the column vectors corresponding to $\prod_{i=1}^{j-1} e^{\gamma_i \mathrm{ad}_{A_i}} A_j$, $j = 2, \ldots, n$, in the same fashion as (11) and then regroup them along the n A_k directions. This can be done explicitly.

$$\begin{aligned}
e^{\gamma_1 \mathrm{ad}_{A_1}} A_2 &\simeq \sum_{r=0}^{n-1} \beta_r^{[1]} c_{1\, p_1}^{k} \ldots c_{1\, 2}^{p_{r-1}} \\
e^{\gamma_1 \mathrm{ad}_{A_1}} e^{\gamma_2 \mathrm{ad}_{A_2}} A_3 &\simeq \sum_{r_1, r_2=0}^{n-1} \beta_{r_1}^{[1]} \beta_{r_2}^{[2]} c_{1\, p_1}^{k} \ldots c_{1\, q}^{p_{r_1-1}} c_{2\, p_1}^{q} \ldots c_{2\, 3}^{p_{r_2-1}} \\
&\vdots \\
\prod_{i=1}^{n-1} e^{\gamma_i \mathrm{ad}_{A_i}} A_n &\simeq \sum_{r_1, \ldots, r_{n-1}=0}^{n-1} \beta_{r_1}^{[1]} \ldots \beta_{r_{n-1}}^{[n-1]} c_{1\, p_1}^{k} \ldots c_{1\, q}^{p_{r_1-1}} c_{2\, p_1}^{q} \ldots \ldots c_{n-2\, q}^{p_{r_{n-2}-1}} c_{n-1\, p_1}^{q} \ldots c_{n-1\, n}^{p_{r_{n-1}-1}}
\end{aligned} \tag{22}$$

The only free index in (22) is k, i.e. each $\prod_{i=1}^{j-1} e^{\gamma_i \mathrm{ad}_{A_i}} A_j$ is an n-dimensional vector as expected. The expression of the $n \times n$ matrix Ξ one obtains by stacking together the columns vectors of (22) is:

$$\Xi = \begin{bmatrix}
1 & \sum_{r=0}^{n-1} \beta_r^{[1]} c_{1\, p_1}^{1} \ldots c_{1\, 2}^{p_{r-1}} & \ldots & \sum_{r_1, \ldots, r_{n-1}=0}^{n-1} \beta_{r_1}^{[1]} \ldots \beta_{r_{n-1}}^{[n-1]} c_{1\, p_1}^{1} \ldots c_{n-1\, n}^{p_{r_{n-1}-1}} \\
0 & \sum_{r=0}^{n-1} \beta_r^{[1]} c_{1\, p_1}^{2} \ldots c_{1\, 2}^{p_{r-1}} & \ldots & \sum_{r_1, \ldots, r_{n-1}=0}^{n-1} \beta_{r_1}^{[1]} \ldots \beta_{r_{n-1}}^{[n-1]} c_{1\, p_1}^{2} \ldots c_{n-1\, n}^{p_{r_{n-1}-1}} \\
\vdots & & \vdots & \\
0 & \sum_{r=0}^{n-1} \beta_r^{[1]} c_{1\, p_1}^{n} \ldots c_{1\, 2}^{p_{r-1}} & \ldots & \sum_{r_1, \ldots, r_{n-1}=0}^{n-1} \beta_{r_1}^{[1]} \ldots \beta_{r_{n-1}}^{[n-1]} c_{1\, p_1}^{n} \ldots c_{n-1\, n}^{p_{r_{n-1}-1}}
\end{bmatrix} \tag{23}$$

5 Example: $SE(3)$

Consider the Lie group $SE(3)$ of rigid body rototranslations in $\mathbb{R}^3$. Using homogeneous coordinates,

$$SE(3) = \left\{ g \in Gl_4(\mathbb{R}),\ \ g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \text{ s.t. } R \in SO(3) \ \text{ and } \ p \in \mathbb{R}^3 \right\}$$

with $SO(3) = \left\{ R \in Gl_3(\mathbb{R}) \text{ s.t. } RR^T = I_3 \ \text{ and } \det R = +1 \right\}$. The Lie algebra of $SE(3)$ is

$$\mathfrak{se}(3) = \left\{ \mathcal{X} \in M_4(\mathbb{R}) \text{ s.t. } \mathcal{X} = \begin{bmatrix} \widehat{\omega}_{\mathcal{X}} & v_{\mathcal{X}} \\ 0 & 0 \end{bmatrix} \ \text{ with } \widehat{\omega}_{\mathcal{X}} \in \mathfrak{so}(3) \ \text{ and } \ v_{\mathcal{X}} \in \mathbb{R}^3 \right\}$$

with $\mathfrak{so}(3) = \left\{ \widehat{\omega}_{\mathcal{X}} \in M_3(\mathbb{R}) \ \text{ s.t. } \widehat{\omega}_{\mathcal{X}}^T = -\widehat{\omega}_{\mathcal{X}} \right\}$ and $\widehat{\cdot} \ : \ \mathbb{R}^3 \to \mathfrak{so}(3)$ such that $\widehat{\omega}_{\mathcal{X}}\sigma = \omega_{\mathcal{X}} \times \sigma$ $\forall\, \sigma \in \mathbb{R}^3$. In the homogeneous representation, a left invariant basis of $\mathfrak{se}(3)$ is given by the 6 matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can interpret A_1 as the infinitesimal generator of the roll angle (around the x axis), A_2 as the pitch generator and A_3 as the yaw generator (axis in the z direction). The nonnull (totally antisymmetric) structure constants $c_{ij}^k = -c_{ji}^k = c_{jk}^i$ corresponding to the basis above are: $c_{12}^3 = c_{15}^6 = c_{42}^6 = c_{34}^5 = 1$

5.1 *Adjoint representation for* $\mathfrak{se}(3)$

For $\mathfrak{se}(3)$, the basis of the adjoint representation is

$$\mathrm{ad}_{A_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathrm{ad}_{A_2} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\mathrm{ad}_{A_3} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathrm{ad}_{A_4} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{ad}_{A_5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathrm{ad}_{A_6} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From eq. (20), the exponentials giving the one parameter groups of automorphisms of $\mathfrak{se}(3)$ computed from the $\beta_j^{[i]}$ are:

$$e^{\gamma_1 \mathrm{ad}_{A_1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\gamma_1 & -\sin\gamma_1 & 0 & 0 & 0 \\ 0 & \sin\gamma_1 & \cos\gamma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\gamma_1 & -\sin\gamma_1 \\ 0 & 0 & 0 & 0 & \sin\gamma_1 & \cos\gamma_1 \end{bmatrix}, \qquad e^{\gamma_4 \mathrm{ad}_{A_4}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\gamma_4 & 0 & 1 & 0 \\ 0 & \gamma_4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\gamma_2 \mathrm{ad}_{A_2}} = \begin{bmatrix} \cos\gamma_2 & 0 & \sin\gamma_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\sin\gamma_2 & 0 & \cos\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\gamma_2 & 0 & \sin\gamma_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\sin\gamma_2 & 0 & \cos\gamma_2 \end{bmatrix}, \qquad e^{\gamma_5 \mathrm{ad}_{A_5}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \gamma_5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\gamma_5 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\gamma_3 \mathrm{ad}_{A_3}} = \begin{bmatrix} \cos\gamma_3 & -\sin\gamma_3 & 0 & 0 & 0 & 0 \\ \sin\gamma_3 & \cos\gamma_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\gamma_3 & -\sin\gamma_3 & 0 \\ 0 & 0 & 0 & \sin\gamma_3 & \cos\gamma_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad e^{\gamma_6 \mathrm{ad}_{A_6}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\gamma_6 & 0 & 1 & 0 & 0 \\ \gamma_6 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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5.2 Wei-Norman formula

Following the canonical ordering given by (4), which at level of angles representations in $\mathfrak{so}(3)$ means choosing XYZ Euler angles (i.e. rotate around the Z axis followed by rotations around the Y and X axes), we obtain the Wei-Norman formula:

$$\Xi = \begin{bmatrix} 1 & 0 & \sin\gamma_2 & 0 & 0 & 0 \\ 0 & \cos\gamma_1 & -\cos\gamma_2\ \sin\gamma_1 & 0 & 0 & 0 \\ 0 & \sin\gamma_1 & \cos\gamma_1\ \cos\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\gamma_2\ \cos\gamma_3 & -\cos\gamma_2\ \sin\gamma_3 & \sin\gamma_2 \\ 0 & 0 & 0 & \xi_{54} & \xi_{55} & \cos\gamma_2\ \sin\gamma_1 \\ 0 & 0 & 0 & \xi_{64} & \xi_{65} & \cos\gamma_1\ \cos\gamma_2 \end{bmatrix}$$

$$\begin{aligned}
\xi_{54} &= \cos\gamma_3\ \sin\gamma_1\ \sin\gamma_2 + \cos\gamma_1\ \sin\gamma_3 \\
\xi_{55} &= \cos\gamma_1\ \cos\gamma_3 - \sin\gamma_1\ \sin\gamma_2\ \sin\gamma_3 \\
\xi_{64} &= -\cos\gamma_1\ \cos\gamma_3\ \sin\gamma_2 + \sin\gamma_1\ \sin\gamma_3 \\
\xi_{65} &= \cos\gamma_3\ \sin\gamma_1 + \cos\gamma_1\ \sin\gamma_2\ \sin\gamma_3
\end{aligned}$$

Lemma 1 implies that the matrix Ξ has to be inverted. This is valid only locally, with domain depending on the choice of the A_i as well as on the ordering in which the products of exponentials is applied in (4). For the case at hand here, the singularities of Ξ correspond to $\det \Xi = \cos\gamma_2 = 0$ i.e. $\gamma_2 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. Out of this set, the inverse of Ξ is given by

$$\Xi^{-1} = \begin{bmatrix} 1 & \sin\gamma_1 \tan\gamma_2 & -\cos\gamma_1 \tan\gamma_2 & 0 & 0 & 0 \\ 0 & \cos\gamma_1 & \sin\gamma_1 & 0 & 0 & 0 \\ 0 & -\sec\gamma_2 \sin\gamma_1 & \cos\gamma_1 \sec\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\gamma_2 \cos\gamma_3 & \chi_{45} & \chi_{46} \\ 0 & 0 & 0 & -\cos\gamma_2 \sin\gamma_3 & \chi_{55} & \chi_{55} \\ 0 & 0 & 0 & \sin\gamma_2 & -\cos\gamma_2 \sin\gamma_1 & \cos\gamma_1 \cos\gamma_2 \end{bmatrix}$$

$$\begin{aligned} \chi_{45} &= \cos\gamma_3 \sin\gamma_1 \sin\gamma_2 + \cos\gamma_1 \sin\gamma_3 \\ \chi_{46} &= -\cos\gamma_1 \cos\gamma_3 \sin\gamma_2 + \sin\gamma_1 \sin\gamma_3 \\ \chi_{55} &= \cos\gamma_1 \cos\gamma_3 - \sin\gamma_1 \sin\gamma_2 \sin\gamma_3 \\ \chi_{55} &= \cos\gamma_3 \sin\gamma_1 + \cos\gamma_1 \sin\gamma_2 \sin\gamma_3 \end{aligned}$$

Changing the ordering in (4), for example using the Euler angles of type ZYZ for the rotation part, the singular points can be moved somewhere else. Thus in this case a complete atlas for $SE(3)$ is obtained, via the exponentials coordinates, by means of two charts only.

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