The reachable set of a linear endogenous switching system

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Abstract

In this work, switching systems are named endogenous when their switching pattern is controllable. Linear endogenous switching systems can be considered as a particular class of bilinear control systems. The key idea is that both types of systems are equivalent to polysystems i.e. to systems whose flow is piecewise smooth. The reachable set of a linear endogenous switching system can be studied consequently. The main result is that, in general, it has the structure of a semigroup, even when the Lie algebra rank condition is satisfied since the logic inputs cannot reverse the direction of the flow. The adaptation of existing controllability criteria for bilinear systems is straightforward.

Keywords: Linear switching systems, bilinear control systems, polysystems, reachability semigroups, controllability.

1 Introduction

Motivated by the need of dealing with physical systems that exhibit a more complicated behavior than those normally described by classical continuous and discrete time domains, hybrid systems are getting very popular nowadays. In particular, there has been a relevant interest in the analysis and synthesis of so-called *switching systems* [10] intended as the simplest class of hybrid systems, as the integral curves of such systems retain continuity although not global smoothness. They can be modeled as a family of plants, all defined in the same domain and such that at each time instant one and only one of them is active, together with a mechanism to govern the switching. This is equivalent to say that the

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vector field that drives the flow of the system is allowed to change in a prescribed family according to a given switching strategy. We concentrate here on a particular class of switching systems: we assume that all modes are linear and that we have complete control over the switching mechanism. We call this class of systems *linear endogenous switching systems*. The assumption of "endogenousness" is crucial as it allows to consider the switching mechanism as a particular control mechanism. Having control parameters is a prerequisite condition to obtain density properties of the trajectories [16] in the reachable set and is not verified when the switching between modes is governed by switching surfaces (we call these *exogenous switching systems*). In this last case, the flow is simply a piecewise smooth ODE, not a control system.

Most of the literature on switching systems has focused on stability problems [2, 5, 1] for the exogenous switching case. Our intention here is to give a characterization of the reachable set of linear endogenous switching systems by relating them to some "classical" work concerning bilinear systems. The key notion is that of *polysystem*, a term used since the seventies [11] to indicate the piecewise smooth approximation of a bilinear system. The suggestion is that switching systems behave like particular types of polysystems and therefore the analogy polysystems - bilinear systems can be used to study controllability properties of switching systems. The polysystem obtained from the linear endogenous switching system is called *SPC polysystem* i.e. *scalar positive polysystem*. It looks like a driftless bilinear control system with control inputs $u_i \in \{0, 1\}$ (positive) such that for all times $\sum_{i=1}^{m} u_i = 1$ (scalar). As a consequence, the trajectories of each mode of the family can flow only along its forward semiorbit. Therefore, unlike driftless bilinear systems, the reachable set of an endogenous switching system has generally only a semigroup structure and the Lie algebra rank condition guarantees only accessibility, not controllability. The formalism we put forward enables the use of the existing Lie algebraic controllability tools developed for bilinear systems (exhaustively surveyed in [8, 13]) to the reachability of linear endogenous switching systems. As an example, the case of a family of switching vector fields living on semisimple Lie algebras is analyzed in detail.

2 Control systems, polysystems and switching systems

In this Section we review the concept of polysystem and use it to relate driftless bilinear control systems to switching systems with controllable logic.

2.1 Bilinear control systems and polysystems

A driftless bilinear control system evolving on a manifold M is expressed as:

$$\dot{x} = \mathcal{F}(x, u) = \sum_{i=1}^{m} u_i A_i x \quad x \in M \tag{1}$$

where $A_i x$ are smooth vector fields, $A_i \in M_n(\mathbb{R})$, $u = (u_1, \ldots, u_m)$ is a control input and belongs to the space $\mathcal{U}$ of all bounded measurable maps defined on any finite interval of the real line of the form $[0,\, T]$, $T \geq 0$.

$$u : [0,\, T] \to U \subseteq \mathbb{R}^m$$

The term polysystem has been used since the seventies [11] to indicate a control system whose inputs are piecewise-constant.

Definition 2.1 *A* dynamical polysystem *on a manifold M is a family*

$$\mathcal{F}_{pc} = \{\mathcal{F}(\cdot,\, u) | u \in \mathcal{U}_{pc}\} \tag{2}$$

of smooth vector fields depending on a piecewise constant parameter u called input.

$\mathcal{U}_{pc}$ is a subclass of the admissible functions $\mathcal{U}_{pc} \subset \mathcal{U}$ such that $u : [0,\, T] \to U$ is piecewise constant. The trajectories of a polysystem are piecewise smooth curves $\gamma : [0,\, T] \to M$ such that, for some partitioning $0 = \theta_0 \leq \theta_1 \leq \ldots \leq \theta_k = T$ of $[0,\, T]$, the restriction of γ to any $[\theta_i,\, \theta_{i+1}]$ is a trajectory of some vector field from $\mathcal{F}_{pc}$. Under some technical assumptions, a control system like (1) can be identified with a polysystem $\mathcal{F}_{pc}$. The technicalities regard the Lebesgue measurability and the local boundedness of the input functions $u(\cdot) \in \mathcal{U}$ and are formalized by Sussmann [15] as:

I. smoothness of the vector fields of $\mathcal{F}_{pc}$

II. U being a metric space.

III. continuity in both x and u of the components of the vector fields together with their derivatives of all orders in x.

If one restricts the set of admissible control inputs $\mathcal{U}$ to the space of piecewise constant functions $\mathcal{U}_{pc}$, then the trajectories of a polysystem coincide with the trajectories of a control system [11]. Actually, the measurability properties of the input u hold if $u \in \mathcal{U}$ is the limit almost everywhere of a sequence of piecewise constant functions. This is, in words, what is normally called the *Approximation lemma* that expands the results on local existence and uniqueness of the solution of ordinary differential equations to more general differential equations depending on parameters, like the control system (1), see [8]. Therefore, all the controllability/accessibility criteria that hold for control systems have an equivalent formulation in terms of polysystems with piecewise constant inputs.

The space of pairs $(u_i, t_i) \in \mathcal{U} \times \mathbb{R}^+$ can be considered as a semigroup under the operation of *concatenation* "$*$". If $u_1,\, u_2 \in \mathcal{U}$, $u_i(\cdot) : [0,\, t_i] \to U$ and $t_i \geq 0$, then the concatenation

$$\begin{aligned} u_1 * u_2(\cdot) \quad &: \quad [0,\, t_1 + t_2] \to U \ \text{ such that} \\ u_1 * u_2(t) &= (u_1,\, t_1) * (u_2,\, t_2) = \begin{cases} u_2(t) & t \in [0,\, t_2] \\ u_1(t) & t \in (t_2,\, t_1 + t_2] \end{cases} \end{aligned} \tag{3}$$

is also in $\mathcal{U}$. Call $\mathcal{U}_*$ the *concatenation semigroup* (or $\mathcal{U}_{pc*}$ for concatenations of piecewise constant input functions). $\mathcal{U}_*$ is also called control semigroup. The concatenation rule $*$ constitutes the multiplicative operation of the semigroup. The "semi" property derives here from the limitation to positive time intervals. This implies that the resulting subset is closed with respect to the multiplication rule but does not admit an inverse. Under some regularity assumptions, like finite number of switching in finite time, the set of piecewise constant inputs $\mathcal{U}_{pc*}$ is then a subsemigroup of $\mathcal{U}_*$.

Call $(u,\, t)_*$ the concatenated sequence of $(u_1,\, t_1)$, $(u_2,\, t_2), \ldots, (u_k,\, t_k)$:

$$(u,\, t)_* = (u_1,\, t_1) * (u_2,\, t_2) * \ldots * (u_k,\, t_k), \qquad u = (u_1,\, u_2, \ldots, u_k), \quad t = (t_1,\, t_2, \ldots, t_k)$$

An autonomous control system is an action Φ of the semigroup $\mathcal{U}_*$ (or $\mathcal{U}_{pc*}$) on M giving the integral curves of the system. In fact, the action

$$\begin{aligned} \Phi \;:\; \mathcal{U}_* \times M &\;\to\; M \\ ((u,\, t)_*,\, x_0) &\;\mapsto\; \Phi((u,\, t)_*, x_0) \end{aligned} \tag{4}$$

maps $\mathcal{U}_*$ into the semigroup T^n of one-to-one continuous homeomorphisms from M to M having flow composition as semigroup operation:

$$\Phi((u,\, t)_*,\, x_0) = \Phi((u_1,\, t_1), \Phi((u_2,\, t_2), \Phi(\ldots\, \Phi((u_k,\, t_k),\, x_0))))$$

The image of $\mathcal{U}_*$ under the map Φ is in general a subsemigroup of T^n For a generic nonlinear system, $\Phi(\cdot)$ does not have a closed form expression. However, it has for a bilinear system like (1) when $u_i \in \mathcal{U}_{pc}$ and it is expressed by means of a product of exponentials.

2.2 SPC polysystem

The rule $*$ will be used in the following to decompose the flow of a polysystem depending on an m-dimensional input vector u into the concatenation of m "elementary" flows depending on a single parameter u_i. If the original nonlinear system is linear in the controls, in $\mathcal{U}_{pc*}$ the concatenation idea can be pushed further: $u_i(t) \in \mathcal{U}_{pc}$ is defined in $[0,\, t_i]$ and has (constant) value $\nu_i \in U$ if and only if $u_i(t) \in \{0,\, 1\}$ and the (time) support is $[0,\, \nu_i\, t_i]$. Since in general $\nu_i \gtrless 0$ then $\nu_i t_i \in \mathbb{R}$ which is not satisfactory if our aim is to have trajectories on $\mathbb{R}^+$ with a proper order relation on it. To fix this we can apply a "cut-off" map before the concatenation operation, limiting the restrained set U to the positive quadrant of $\mathbb{R}^m$. These requirements are described in the following in terms of a cascade of maps in input space.

Consider an interval, without loss of generality $[0,\, \tau]$, in which the input $u \in \mathcal{U}_{pc}$ remains constant. Since $U \in \mathbb{R}^m$, call e_i the i-th element of the standard basis of $\mathbb{R}^m$. In $[0,\, \tau]$, the first map we apply "$S(\cdot)$" is meant to decompose $u = [u_1\, u_2 \ldots u_i \ldots u_m]^T$, $u_i = \langle u,\, \mathrm{e}_i \rangle$, into the sequence of m inputs ν_i depending on a single parameter: $\nu_i = [0\, 0 \ldots u_i \ldots 0]^T$.

- sequentialization S: transforms an m-dimensional input into a sequence of m scalar inputs along each element e_i of the standard basis of $\mathbb{R}^m$

$$\begin{aligned} S \;:\; \mathcal{U}_{pc} &\;\to\; \mathcal{U}_{pc_1} \times \mathcal{U}_{pc_2} \times \ldots \times \mathcal{U}_{pc_m} \\ u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} &\;\mapsto\; \begin{bmatrix} u_1 & 0 & & 0 \\ 0 & u_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & u_m \end{bmatrix} = \begin{bmatrix} \langle u, \mathrm{e}_1 \rangle \mathrm{e}_1 \ldots \langle u, \mathrm{e}_m \rangle \mathrm{e}_m \end{bmatrix} = \begin{bmatrix} \nu_1 & \ldots & \nu_m \end{bmatrix} \end{aligned} \tag{5}$$

where $\langle\cdot,\,\cdot\rangle$ is the inner product in $\mathbb{R}^m$. The need of moving only along the positive direction of the orbit of each vector field means we have to use only positive controls, which are selected by the "cut-off" map "$T(\cdot)$"

- truncation T: to each ν_i apply the following:

$$\begin{array}{rcl} T\ :\ \mathcal{U}_{pc_i} & \to & \mathcal{U}^+_{pc_i} = \left\{\nu_i\ :\ [0,\,t_i] \to U^+ = U \cap \mathbb{R}^{m+}\right\} \\ \nu_i = \langle u, \mathrm{e}_i\rangle \mathrm{e}_i = u_i \mathrm{e}_i & \mapsto & = \langle u(u_i > 0), \mathrm{e}_i\rangle \mathrm{e}_i = u_i(u_i > 0)\mathrm{e}_i \end{array}$$

where $u(v > 0) = u$ if $v > 0$ and 0 otherwise. If the system is linear in the control and $u = \text{const}$ then its integral is linear in both u and time. Restricting the input to take values in $\{0,\,1\}$, the same integral curve is obtained by altering the time support. Call "$M(\cdot)$" such a map (that "flattens" all the inputs to 1 by modulating with respect to the time support):

- modulation in the time support M: call $\mathcal{V}_i = \{0,\,\mathrm{e}_i\}$

$$\begin{array}{rcl} M\ :\ \mathcal{U}^+_{pc_i} \times \mathbb{R}^+ & \to & \mathcal{V}_i \times \mathbb{R}^+ \\ (\nu_i(u_i > 0),\,t) = (u_i(u_i > 0)\mathrm{e}_i,\,t) & \mapsto & (\mathrm{e}_i,\,u_i(u_i > 0)t) \end{array}$$

To the sequence obtained by the previous maps we can apply the pairwise operation of Section 2.1:

- concatenation $*$:

$$\begin{array}{rcl} *\ :\ \left(\mathcal{V}_i \times \mathbb{R}^+\right) \times \left(\mathcal{V}_j \times \mathbb{R}^+\right) & \to & (\mathcal{V}_i \times \mathcal{V}_j) \times \mathbb{R}^+ = \{0,\,\mathrm{e}_i,\,\mathrm{e}_j\} \times \mathbb{R}^+ \\ (\mathrm{e}_i,\,u_i(u_i > 0)t_i),\,(\mathrm{e}_j,\,u_j(u_j > 0)t_j) & \mapsto & \begin{cases} \mathrm{e}_j & t \in [0,\,u_j(u_j > 0)t_j) \\ \mathrm{e}_i & t \in [u_j(u_j > 0)t_j,\ u_j(u_j > 0)t_j + u_i(u_i > 0)t_i) \end{cases} \end{array}$$

For sake of conciseness, we call the composition of the four maps above (transforming the control in scalar and positive unitary) the *SPC map* "$\circledast$"

$$\circledast \triangleq * \circ M \circ T \circ S$$

Calling $\mathcal{V} \triangleq \{0,\,\mathrm{e}_1,\,\mathrm{e}_2,\ldots,\mathrm{e}_m\}$, in correspondence of a control input $u \in \mathcal{U}_{pc}$ of constant value in $[0,\,\tau]$ we have the following piecewise constant input $(u,\,\tau)_\circledast$ in $[0,\,\sum_{i=1}^m u_i(u_i > 0)\tau]$:

$$\begin{aligned} \circledast\ :\ \mathcal{U}_{pc} \times \mathbb{R}^+ \ &\to\ \mathcal{V} \times \mathbb{R}^+ \\ (u,\,\tau)\ &\mapsto\ \begin{cases} \mathrm{e}_m & t \in [0,\,u_m(u_m > 0)\tau) \\ \mathrm{e}_{m-1} & t \in [u_m(u_m > 0)\tau,\ u_m(u_m > 0)\tau + u_{m-1}(u_{m-1} > 0)\tau) \\ \vdots & \\ \mathrm{e}_1 & t \in [\sum_{i=2}^m u_i(u_i > 0)\tau,\,\sum_{i=1}^m u_i(u_i > 0)\tau] \end{cases} \\ &= (\mathrm{e}_1,\ ,u_1(u_1 > 0)\tau) * (\mathrm{e}_2,\ ,u_2(u_2 > 0)\tau) * \ldots * (\mathrm{e}_m,\ ,u_m(u_m > 0)\tau) \\ &= (u,\,\tau)_\circledast \end{aligned} \tag{6}$$

The order in which the inputs ν_i appear depend on the sequentialization (5). However, since the underlying space $\mathbb{R}^m$ commutes, any order is equivalent in $\circledast$. This is true only in the input space, when the flow action (4) is applied, the commutativity depends on the vector fields of $\mathcal{F}$.

In exactly the same way as in Section 2.1, the map $*$ can be used to concatenate pieces of SPC inputs.

Proposition 2.1 *The map $\circledast$ preserves the structure of semigroup of the concatenated input space $\mathcal{U}_{pc*}$.*

Proof Associativity of the concatenation $*$ is maintained by $\circledast$ as can be verified by the straightforward computation:

$$\left((u^1,\, \tau^1)_\circledast * (u^2,\, \tau^2)_\circledast\right) * (u^3,\, \tau^3)_\circledast = (u^1,\, \tau^1)_\circledast * \left((u^2,\, \tau^2)_\circledast * (u^3,\, \tau^3)_\circledast\right)$$

$\square$

The space of admissible inputs is

$$\mathcal{V}_{sp} = \left\{ u = (u_1,\, u_2, \ldots, u_m) \mid u_i \in \{0,\, 1\} \text{ and } \sum_{i=1}^m u_i = 1 \right\} \tag{7}$$

The difference with respect to a control set of bang-bang type like

$$\mathcal{V}_s = \left\{ u = (u_1,\, u_2, \ldots, u_m) \mid u_i \in \{-1,\, 0,\, 1\} \text{ and } \sum_{i=1}^m |u_i| = 1 \right\} \tag{8}$$

is that the negative values of the input are missing because of the truncation map mentioned above. The most immediate consequence is that, since the time span is $\mathbb{R}^+$, the integral curve of the polysystem

$$\begin{array}{rccc} \Phi \,:\, & \mathcal{V}_{sp} \times M & \to & M \\ & ((u,\, \tau)_\circledast,\, x_0) & \mapsto & \Phi\left((u,\, \tau)_\circledast, x_0\right) \end{array} \tag{9}$$

are forced to move only in the "positive direction" of the vector fields. Each piece of flow corresponding to some negative input disappear i.e. it is mapped by Φ to the neutral element of the flow composition:

$$\Phi((0,\, t),\, x_0) = \Phi((1,\, 0),\, x_0) = x_0$$

We call the polysystem under the concatenation action "$\circledast$" an *SPC polysystem* i.e. *polysystem with scalar positive (unitary) controls.*

Definition 2.2 *The* SPC polysystem $\mathcal{F}_{sp}$ *associated with the bilinear system* (1) *is a dynamical polysystem whose input u takes values in $\mathcal{V}_{sp}$:*

$$\mathcal{F}_{sp} = \{\mathcal{F}(\cdot,\, u) | u \in \mathcal{V}_{sp}\} \tag{10}$$

2.3 Switching systems

Unlike hybrid systems, where the trajectories are allowed to have discontinuous jumps due to some change in either the continuous or the discrete dynamics of the system, the term switching system is used to describe systems in which the change of some operative mode maintains the continuity of the flow of the solution even though not its smoothness. A number of different multi-modal systems can be classified as switching systems. See [5, 2, 17, 18] for some examples of formulations. The bottom line of all the different formulations is that a switching system is composed of a family of different (smooth) dynamic modes such that the switching pattern gives continuous, piecewise smooth trajectories. Moreover, we assume that one and only one mode is active at each time instant.

The different switching schemes can be classified in the two categories: *endogenous switching* (or *controlled switching* or *switching-on-time*) and *exogenous switching* (or *autonomous switching* or *switching-on-state*). The endogenous switching is the simplest of the two because it involves only changes in the tangent space (the switching from one element to another one of the family of vector fields can be decided arbitrarily as well as the instant of switching) without need to check what happens on the flow of the solution. The exogenous switching is more complicated: in fact it requires to know exactly the integral curves of the system in order to decide when to pass from a dynamic mode to another one and can be thought of as a feedback loop in which the switching logic is governed by a partition of the configuration space. Similarly, the endogenous scheme can be thought of as the open loop version of the same system for a special class of control inputs.

The following definition is on the integral curves of a switching system and holds regardless of the switching scheme used.

Definition 2.3 *A* switching system *on M is a collection of smooth vector fields $\mathcal{F}_{sw} = \{X_i | i \in \mathcal{K}\}$, with $\mathcal{K}$ some index set, characterized by integral curves $\gamma : [0, T] \to M$ that are continuous and piecewise smooth i.e. they admit a partition $0 = \theta_0 \leq \theta_1 \leq \ldots \leq \theta_k = T$ of $[0, T]$ such that the restriction of γ to the open interval (θ_i, θ_{i+1}) is differentiable and $\dot{\gamma} = X_i(\gamma(t))$ for some $i \in \mathcal{K}$.*

In the following, we will consider only endogenous switching systems.

2.4 Transition matrix Lie group of a bilinear systems on $\mathbb{R}^n \setminus \{0\}$

This paragraph is only meant to enable us to use the terminology of group (of diffeomorphisms) for the reachable set of a bilinear system with inputs in $\mathcal{U}_{pc}$ or in $\mathcal{V}_s$ (as opposite to the weaker structure of semigroup mentioned above).

The class of bilinear systems (and more in general of affine in control systems [9]) lives on homogeneous spaces that are subordinated to a Lie group action. In particular, if $M = \mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}$ (we consider the punctured euclidean space as the origin is an isolated equilibrium point), the group of automorphisms of $\mathbb{R}^n$, $GL_n^+(\mathbb{R})$ (the connected component of $GL_n(\mathbb{R})$ containing the identity) defines a linear action on it:

$$\theta_g(x) = gx \qquad g \in GL_n^+(\mathbb{R}), \quad x \in \mathbb{R}^n_0$$

The vector fields of $\mathcal{F}$: $A_i x$, $A_i \in \mathfrak{gl}_n$ can be lifted for example to right invariant vector fields $A_i g$ on $T_g GL_n(\mathbb{R})$ for each $g \in GL_n^+(\mathbb{R})$. For the bilinear system (1), a necessary and sufficient condition for controllability in $\mathbb{R}_0^n$ is the transitivity of the action of the group of automorphisms for all $x_0 \in \mathbb{R}_0^n$, see [4]. If $\mathrm{Lie}(\mathcal{F})$ is a proper subgroup of $\mathfrak{gl}_n$ the isotropy subgroup $H = \{g \in G \mid \theta_g(x) = x\}$ is nontrivial and $\mathcal{F}$ can be lifted to GL_n/H. In the following we consider $G = GL_n/H$ and $\mathfrak{g}$ the Lie algebra of G. The infinitesimal generators of the matrix representation of $\mathfrak{g}$ are the A_i. The transitivity property can be formulated in terms of the well-known Lie algebraic rank condition (LARC)

$$\mathrm{rank}\,(\mathrm{Lie}(\mathrm{A_i})) = \mathrm{dim} Gx = n \qquad x \in \mathbb{R}_0^n$$

Call Γ the family of vector fields of $\mathcal{F}$ lifted to G. To the bilinear system (1) we can associate a matrix bilinear system having the right invariant representation:

$$\begin{aligned} \dot{g}(t) &= \sum_{i=1}^{m} u_i(t) A_i g(t) \qquad g \in G \quad u \in \mathcal{U} \\ g(0) &= e \end{aligned} \tag{11}$$

The matrix g is normally called the transition matrix of x and it represents the evolution of the system (1) from n independent initial conditions.

If the system has a drift, the LARC is only a necessary condition for controllability.

2.5 The SPC polysystem of a bilinear system

Consider the bilinear system (1). Locally its solution can be expressed via the polysystem in terms of a single exponential. In $[0, \tau]$, applying the constant input $u = [u_1 \dots u_m] = [\langle u, \mathrm{e}_1\rangle \dots \langle u, \mathrm{e}_m\rangle]$:

$$x(\tau) = \Phi\left((u,\, \tau), x_0\right) = e^{\sum_{i=1}^{m} A_i u_i \tau} x_0 \tag{12}$$

The limitation to positive unitary controls is such that for each $u_i = \langle u, \mathrm{e}_i\rangle$, $i = 1, \dots, m$, only $\langle u(u_i > 0), \mathrm{e}_i\rangle$ is considered. Associating each element e_i of the standard basis of $\mathbb{R}^m$ with the infinitesimal generator $A_i \in \mathfrak{g}$, the exponential map from $\mathfrak{g}$ to G induces a transition matrix corresponding to a one-parameter flow along A_i

$$\begin{aligned} \Phi\left((u_i,\, \tau), e\right) = \Phi\left((\mathrm{e}_i,\, \langle u(u_i > 0), \mathrm{e}_i\rangle\tau),\, e\right) \;:\; \mathbb{R}^+ &\rightarrow G \\ t &\mapsto e^{A_i t} = e^{A_i u_i (u_i > 0)\tau} \end{aligned}$$

which is a one-parameter subsemigroup of G ("half" orbit). The truncated single exponential (12) is then

$$x(\tau) = e^{\sum_{i=1}^{m} A_i t_i} x_0 \tag{13}$$

where $t_i = u_i(u_i > 0)\tau \geq 0$, $i = 1, \dots, m$. When the input concatenation operation $\circledast$ is applied to (1)

$$\begin{aligned} x(t) &= \Phi\left((u,\, \tau)_{\circledast},\, x_0\right) \\ &= \Phi\left((\mathrm{e}_1,\, t_1),\, \Phi\left((\mathrm{e}_2,\, t_2),\, \dots \Phi\left((\mathrm{e}_m,\, t_m)\right)\right)\right) x_0 \qquad t = \sum_{i=1}^{m} u_i(u_i > 0)\tau \end{aligned} \tag{14}$$

The flow concatenation (14) admits an explicit expression in terms of product of exponentials:

$$x(t) = e^{A_1 t_1} e^{A_2 t_2} \dots e^{A_m t_m} x_0 \tag{15}$$

For the matrix system (11), both the exponentials (12) and (15) represent integral curves of (11) starting from the origin of G called canonical coordinates of the first and second kind respectively. In practice, the concatenation is nothing but a way to transform any smooth trajectory into an arcwise connected one (which is "more evidently" accessible). For trajectories that span the whole $\mathfrak{g}$ the corresponding concatenation gives a so-called *normally accessible* trajectory. The order of application here follows the rule (3), but such choice is completely arbitrary, unless a switching pattern is prespecified. The change of order in the product of exponentials can be obtained by repeated application of the Campbell-Baker-Hausdorff formula (expressing how much the brackets fail to commute over the exponential), see [3].

2.6 Linear endogenous switching systems and SPC polysystems

Recall that in the endogenous switching case one assumes to have complete control over:

I. time of switch

II. switching pattern

After all of the previous discussion, the following theorem is almost a tautology.

Theorem 2.1 *Consider an endogenous switching system $\mathcal{F}_{sw}$ on M given by the same family of linear vector fields of the drift-free bilinear system* (1)*: $X_i(x) = A_i x,\ i = 1, \dots, m,\ A_i \in M_n(\mathbb{R})$. The trajectories of such switching system coincide with those of the SPC polysystem $\mathcal{F}_{sp}$ obtained from* (1)*.*

Proof The trajectories of the linear switching system with

$$\begin{cases} \text{switching pattern } k_1 \to k_2 \to \dots \to k_p \\ \text{switches at times } t_{k_1},\ t_{k_2}, \dots, t_{k_p} \quad \in \mathbb{R}^+ \end{cases} \tag{16}$$

initialized at $x_0 \in M$ look like composition of exponentials:

$$x(t) = e^{A_{k_1} t_{k_1}} e^{A_{k_2} t_{k_2}} \dots e^{A_{k_p} t_{k_p}} x_0 \qquad t = t_{k_1} + t_{k_2} + \dots t_{k_p} \geq 0$$

Since the switching system and the SPC polysystem have the same family of generators, any of the sequences (16) can be replied by the SPC polysystem via a suitable sequence of $*$ and $\circledast$ operations. The converse implication follows from the same argument as both sequences $k_1 \to k_2 \to \dots \to k_p$ and $t_{k_1},\ t_{k_2}, \dots, t_{k_p}$ are controllable and can be chosen so as to match a given input concatenation pattern in the bilinear system. $\square$

3 Reachability semigroups for an SPC polysystem

Given $x_0 \in M$, let us call $\mathcal{R}_{\mathcal{F}}(x_0, T)$ the *reachable set from x_0 at time $T > 0$* for the system $\mathcal{F}$:

$$\mathcal{R}_{\mathcal{F}}(x_0, T) = \{x \in M \text{ s.t. } \exists \text{ an input in } \mathcal{U}\ u : [0, T] \to U \text{ such that the evolution of (1) satisfies } x(0) = x_0 \text{ and } x(T) = x\}$$

and the *reachable set from x_0 in time not greater than T*

$$\mathcal{R}_{\mathcal{F}}(x_0, \leq T) = \cup_{0 \leq t \leq T} \mathcal{R}_{\mathcal{F}}(x_0, t)$$

The system $\mathcal{F}$ is *small-time locally controllable* at x_0 if $\mathcal{R}_{\mathcal{F}}(x_0, \leq T)$ contains a non-empty open subset of M for all $T \geq 0$ and for all neighborhoods of x_0 and x_0 belongs to the interior of this subset. The existence of a nonempty interior in M is referred to as accessibility property and only for drift-free systems it corresponds to controllability.

Global controllability from x_0 is equivalent to have

$$\mathcal{R}_{\mathcal{F}}(x_0) = \cup_{0 \leq t \leq \infty} \mathcal{R}_{\mathcal{F}}(x_0, t) = M$$

The system is said *globally controllable* if it is globally controllable from each $x_0 \in M$.

For a system $\mathcal{F}$, unlike the reachable set $\mathcal{R}_{\mathcal{F}}(x_0)$ which accounts only for the positive time evolution of the trajectories of $\mathcal{F}$, an orbit $\mathcal{O}_{\mathcal{F}}(x_0)$ requires to consider complete vector fields, i.e. defined on the whole time axis:

$$\mathcal{O}_{\mathcal{F}}(x_0) = \cup_{t \in \mathbb{R}} \{x \in M \text{ s.t. } \exists \text{ an input in } \mathcal{U} \text{ such that the evolution of (1) satisfies } x(0) = x_0 \text{ and } x(t) = x \quad t \in \mathbb{R}\}$$

For both $\mathcal{R}$ and $\mathcal{O}$, right invariance of (11) implies $\mathcal{R}_{\mathcal{F}}(x_0) = \mathcal{R}_{\Gamma}(e)x_0$ and $\mathcal{O}_{\mathcal{F}}(x_0) = \mathcal{O}_{\Gamma}(e)x_0$ and thus we can talk indifferently of attainable sets from e of Γ on G and of attainable sets of $\mathcal{F}$ from any $x_0 \in M$, $x_0 \neq 0$, see [12]. Call $\mathcal{R}_{\mathcal{F}_{pc}}$ and $\mathcal{R}_{\mathcal{F}_{sp}}$ the attainable sets of the polysystem and of the SPC polysystem associated with (1).

When considering right invariant systems on G, an elementary basic necessary condition for global controllability of any switching scheme is that the Lie group to which the system $\mathcal{F}$ can be lifted has to be connected. For example if we have $\mathcal{F} \in \mathfrak{gl}_n(\mathbb{R})$ then we have to consider only the component $GL_n^+(\mathbb{R})$ of the general linear group.

For the system (1), collecting together the main (well-known) results on controllability we have:

Theorem 3.1 *The bilinear driftless system* (1)

(a) is globally controllable from a point $x_0 \in \mathbb{R}_0^n$ if and only if any of the three equivalent properties holds:

I. G transitive

II. $\operatorname{rank}(\operatorname{Lie}(\mathcal{F})) = n$

III. $x_0 \in \operatorname{int}(\mathcal{R}_{\mathcal{F}})$.

(b) Under the assumption (a), $\mathcal{R}_\Gamma(e) = \mathcal{O}_\Gamma(e)$ is a group and the associated polysystem retains the same attainable set: $\mathcal{R}_{\mathcal{F}_{pc}} = \mathcal{R}_{\mathcal{F}}$.

Proof All three items of (a) are classical results, see for example [8, 4, 12]. The first two were already used above; here we comment only on the item 3. Since $\mathcal{R}_{\mathcal{F}}(x_0) = \mathcal{R}_\Gamma(e)x_0$ it is enough to show that Γ is controllable on a connected G if and only if e is contained in an interior of $\mathcal{R}_\Gamma(e)$ which is a classical result for control systems on Lie groups (see [8] p. 154 or [13] p. 9). Roughly speaking, if e lies on the boundary of the reachable set then there exists "forbidden" directions from e and therefore the reachable set is not a group. On the other hand, controllability implies that the orbit from e is the entire group. Also item (b) is one of the basic results in control theory, deriving from the orbit theorem. See for example the book [8], Chapter 2. Its extension to the polysystem is done via the approximation lemma mentioned in Section 2.1. □

In the more general case of a system with a drift term $\mathcal{O}_{\mathcal{F}}(x_0)$ is a subgroup but $\mathcal{R}_{\mathcal{F}}(x_0)$ is only a subsemigroup. The conditions in part *(a)* of Theorem 3.1 become necessary and sufficient conditions for accessibility. Necessary and sufficient conditions for controllability are in general not known; for example the small time local controllability mentioned above provides a sufficient but not necessary condition for controllability.

The final point reached by the original polysystem and by the SPC polysystem after the concatenation action are in general different. For the polysystem $\mathcal{F}_{pc}$ with a noncompact transition matrix Lie group, the limitation to a control set like (7) normally forbids to get a reachable set which is a group, while a symmetric control set like (8) allows to span the entire orbits achieving controllability via some form of bang-bang control, when the LARC is satisfied. This is equivalent to say that the reachable sets of the polysystem and of the SPC polysystem obtained from it are different. In fact, excluding the compact case, it is not in general possible to give a group structure to $\mathcal{R}_{\mathcal{F}_{sp}}$, not even if the system is driftless and the LARC is satisfied. Considering the convex hull "co" of the family $\mathcal{F}$ of control vector fields:

$$\mathrm{co}\left(\mathcal{F}(x_0,\, \mathcal{V}_{sp})\right) = \mathcal{F}\left((x_0,\, \mathrm{co}(\mathcal{V}_{sp})\right)$$

Since the convex hull of the control set is not a neighborhood of x_0, the input vector fields are not complete. Therefore, unlike for the original polysystem, the accessibility problem for the SPC polysystem obtained by concatenation is a semigroup problem also in the drift-free case.

Theorem 3.2 *Given the bilinear driftless system* (1)*, the reachable set $\mathcal{R}_{\mathcal{F}_{sp}}$ of the associated SPC polysystem $\mathcal{F}_{sp}$ is a subsemigroup of $\mathcal{R}_{\mathcal{F}}$: $\mathcal{R}_{\mathcal{F}_{sp}} \subseteq \mathcal{R}_{\mathcal{F}}$.*

Proof By Proposition 2.1, the map $\circledast$ does not spoil the semigroup structure of the concatenation. Furthermore, associativity with respect to the flow operation for the SPC polysystem can be checked directly:

$$\Phi\left(\left((u^1,\, \tau^1)_\circledast * (u^2,\, \tau^2)_\circledast\right) * (u^3,\, \tau^3)_\circledast\right) = \Phi\left((u^1,\, \tau^1)_\circledast * \left((u^2,\, \tau^2)_\circledast * (u^3,\, \tau^3)_\circledast\right)\right)$$

Both sides are ordinary products of exponentials, some of which might have zero exponent. □

Corollary 3.1 *An endogenous switching system $\mathcal{F}_{sw}$ being a SPC polysystem, its reachable set $\mathcal{R}_{\mathcal{F}_{sw}}$ is a semigroup equal to $\mathcal{R}_{\mathcal{F}_{sp}}$.*

The following comment highlights the difference between the bilinear system (1) and the corresponding switching system:

Remark: *The Lie algebraic properties of the endogenous switching system are uniquely defined by the vector fields A_i and not by the control parameters.*

This is clear if the control set is $\mathcal{V}_s$: $u_i \in \{-1, 0, 1\}$ are all is needed to have complete input vector fields and span$\{X_i(u_i,t),\ X_i \in \mathcal{F},\ u_i \in \mathcal{V}_s,\ t \in \mathbb{R}^+\}$ is equivalent to Lie$(\mathcal{F})$. Admitting any larger control set does not increase the dimension of the linear span of the X_i. When the control set is $\mathcal{V}_{sp}$, we have to consider a tangent object which is a convex cone in the vector space $\mathfrak{g}$. The Lie algebraic properties of the family of vector fields are untouched (although there is a lot to say concerning the relation between the convex cone and the Lie algebra that contains it, see the book [7] for an overview).

Since for all times at most one of the vector fields is active, the SPC polysystem is always driven by a scalar control, therefore properties like exact-time controllability or strong controllability differ from their counterparts on the corresponding bilinear system and will not be considered here.

In summary then, the main result of the paper can be stated as follows:

Theorem 3.3 *If the driftless system* (1) *satisfies the LARC condition, then*

I. as a bilinear system (with space of admissible input functions either $\mathcal{U}$ or $\mathcal{U}_{pc}$ or $\mathcal{V}_s$)

$$\mathcal{R}_{\mathcal{F}} = \mathcal{R}_{\mathcal{F}_{pc}} = M$$

II. as a switching system (with space of admissible input functions $\mathcal{V}_{sp}$)

$$\mathcal{R}_{\mathcal{F}_{sw}} \subseteq M$$

In fact, if instead of the entire Lie algebra of complete vector fields one considers only cones stable under multiplications by nonnegative coefficients as those generated by our endogenous switching systems along the inhomogeneous directions, then the LARC condition will only guarantee $\mathcal{R}_{\Gamma_{sw}}$ such that $\mathcal{R}_{\mathcal{F}_{sw}} = \mathcal{R}_{\Gamma_{sw}} x_0$ to be a maximal semigroup i.e. a semigroup of G which is not properly contained in any other proper semigroup of G.

4 Homogeneous switching systems

When a vector field $A \in \mathcal{F}_{sw}$ is such that also $-A \in \mathcal{F}_{sw}$, then the orbit of A is a full one-parameter group and A can be considered a complete vector field. For analogy with the ordinary control systems, we will call such A a *homogeneous vector field*. Thus, even with the control set $\mathcal{V}_{sp}$, the vector space structure can be recovered with a double number of infinitesimal generators. We will call *homogeneous switching system* a system $\mathcal{F}_{sw}$ such that all $A \in \mathcal{F}_{sw}$ are homogeneous.

In the case of a homogeneous family of linear vector fields, the trajectories of the switching system are exactly those of the scalar polysystem of the corresponding bilinear system.

Proposition 4.1 *The reachable set $\mathcal{R}_{\mathcal{F}_{sp}}$ of an endogenous homogeneous switching system is a group.*

Proof According to the definition above, all vector fields are complete, therefore the reachable set is a union of orbits. $\square$

Corollary 4.1 *An endogenous homogeneous linear switching system $\mathcal{F}_{sp}$ is controllable if and only if it satisfies the LARC condition.*

In the following, we subdivide $\mathcal{F}_{sw}$ in homogeneous and inhomogeneous vector fields, labeled respectively by the index sets $\mathcal{K}_h$ and $\mathcal{K}_i$:

$$\mathcal{F}_{sw} = \{X_j \,|\, j \in \mathcal{K}\} = \{X_j \,|\, j \in \mathcal{K}_h\} \cup \{X_j \,|\, j \in \mathcal{K}_i\} = \mathcal{F}_{sw_h} \cup \mathcal{F}_{sw_i}$$

Then $A \in \mathcal{F}_{sw_h}$ is a complete vector field while $B \in \mathcal{F}_{sw_i}$ is not, and if we call $\mathrm{card}(\cdot)$ the cardinality of a set (i.e. the number of elements in the set)

$$2\mathrm{card}(\mathcal{K}_h) + \mathrm{card}(\mathcal{K}_i) = \mathrm{card}(\mathcal{K})$$

5 Application: controllability of linear endogenous systems on semisimple Lie algebras

For details on the (standard) notions on Lie algebras used in this Section the reader is addressed to the literature, eg. [6, 14]. The simplest possible case of linear endogenous switching system one can encounter is that of a Lie algebra $\mathfrak{g}$ generated by the family $\mathcal{F}_{sw}$, $\mathfrak{g} = \mathrm{Lie}(\mathcal{F}_{sw})$, which is abelian. A Lie algebra is said abelian if all the elements of $\mathfrak{g}$ commute with respect to the Lie bracket operation. For an endogenous switching system with abelian Lie algebra, controllability is never an issue: if $\mathcal{F}_{sw} = \mathfrak{g}$, then controllability (both local and global) is assured; if not, there is no way to move along the missing directions.

Here we analyze in detail the more complicated case of semisimple Lie algebras. From the Levi decomposition, every Lie algebra (and thus Lie algebras of linear vector fields) can be decomposed into a semidirect product of a semisimple Lie algebra and a solvable one. A given Lie algebra $\mathfrak{g}$ is said *semisimple* if it contains no abelian ideals other than 0. Given any pair of vector fields A, B in a semisimple Lie algebra $\mathfrak{g}$, we have the following generic result:

Lemma 5.1 (Theorem 12, Ch.6 of [8]) *The set of pairs $A, B \in \mathfrak{g}$ such that* $\mathrm{Lie}(A, B) = \mathfrak{g}$ *is open and dense in $\mathfrak{g}$ semisimple.*

From Theorem 3.3, Part I, this is enough to affirm that the driftless system $\mathcal{F} = \{A, B\}$ (or the corresponding $\mathcal{F}_{pc}$) is globally controllable, while it does not guarantee controllablity on the corresponding switching system. However, it is necessary to distinguish between compact and noncompact semisimple Lie algebras. Semisimple Lie algebras are classified according to the eigenvalues of the Killing form i.e. the symmetric bilinear form $K(A, B) = \mathrm{tr}(\mathrm{adA} \cdot \mathrm{ad}B)$.

$K(\cdot,\cdot)$ is always nondegenerate on a semisimple Lie algebra; if all its eigenvalues are negative, then the Lie algebra is *compact* (ex. $\mathfrak{so}(n)$), otherwise it is *noncompact* (ex. $\mathfrak{so}(n,m)$ or $\mathfrak{sl}(n)$).

Adapting the result on controllability of semisimple Lie algebras, see [8, 13] to our situation, we obtain sufficient conditions for global controllability of endogenous switching systems on semisimple Lie algebras based only on counting the vector fields on $\mathcal{K}$.

Theorem 5.1 *Given $\mathcal{F}_{sw}$ such that $\mathrm{Lie}(\mathcal{F}_{sw})$ is semisimple, we have:*

I. a sufficient condition for global controllablity of $\mathcal{F}_{sw}$ is that $\mathrm{card}(\mathcal{K}_h) \geq 2$;

II. if $\mathrm{Lie}(\mathcal{F}_{sw})$ is compact, then a sufficient condition for global controllablity of $\mathcal{F}_{sw}$ is that $\mathrm{card}(\mathcal{K}_h) + \mathrm{card}(\mathcal{K}_i) \geq 2$;

Both conditions are generic, i.e. hold in an open and dense set of the Lie algebra.

Proof Condition I follows from Lemma 5.1 and then Part I of Theorem 3.3, i.e. it corresponds to having two complete vector fields which are generically generating in a semisimple Lie algebra. For the compact case, the weaker Condition II is a consequence of the following fact (see [8], Ch. 6 Lemma 1) which holds for any vector field A in a compact Lie algebra:

$$\mathrm{cl}\left(e^{tA},\ t<0\right) \subset \mathrm{cl}\left(e^{tA},\ t \geqslant 0\right)$$

where $\mathrm{cl}(\cdot)$ means closure. In fact, a compact Lie algebra does not admit semigroups [7]. Thus also inhomogeneous vector fields generate subgroups in this case and a lower number of vector fields in $\mathcal{K}$ is needed for the sufficient condition to hold. □

In practice, while the condition $\mathrm{card}(\mathcal{K}_i) = 2$ (i.e. $\mathcal{K}$ is a collection of two vector fields) is generically enough to guarantee controllability of $\mathfrak{g}$ compact, the stronger condition $\mathrm{card}(\mathcal{K}_h) = 2$ (i.e. $\mathcal{K}$ must contain two pairs of vector fields of opposite signs) is needed for the noncompact case. It must be noticed that Condition I above can be weakened to $\mathrm{card}(\mathcal{K}_i) = \mathrm{card}(\mathcal{K}_h) = 1$ by studying the root system of $\mathfrak{g}$, see [13] for details. Notice further that small time local controllability requires $\mathrm{card}(\mathcal{K}_h) = 2$ regardless of the character of the semisimple Lie algebra.

6 Conclusion

A switching system can be seen as a collection of differential equations plus a logic mechanism that allows to switch between them. For endogenous switching systems, the reachability problem is well-posed and a characterization of the reachable set can be done by using the analogy endogenous switching system - polysystem - control system. Since linear endogenous switching systems correspond to driftless bilinear control systems with positive controls, well-known controllability conditions of the bilinear systems can be reframed to our situation. The main result is that the reachable set is normally only a semigroup since the switching logic allows to pass from a vector field to another but not to reverse the direction of motion. Therefore the Lie algebra rank condition does not assure controllability of the system but only the weaker accessibility property, like in the case of bilinear systems with drift. Most of the existing criteria for bilinear systems with drift can be easily adapted to reachability analysis of switching systems.

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