## Involutive flows and discretization errors for nonlinear driftless control systems<sup>\*</sup>

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#### Abstract

In a continuous-time nonlinear driftless control system, an involutive flow is a composition of input profiles that does not excite any Lie bracket. Such flow composition is trivial, as it corresponds to a "forth and back" cyclic motion obtained rewinding the system along the same path. The aim of this paper is to show that, on the contrary, when a (nonexact) discretization of the nonlinear driftless control system is steered along the same trivial input path, it produces a net motion, which is related to the gap between the discretization used and the exact discretization given by a Taylor expansion. These violations of involutivity can be used to provide an estimate of the local truncation error of numerical integration schemes. In the special case in which the state of the driftless control system admits a splitting into shape and phase variables, our result corresponds to saying that the geometric phases of the discretization need not obey an area rule, i.e., even zero-area cycles in shape space can lead to nontrivial geometric phases.

**Keywords:** Discrete-time nonlinear control systems; nonintegrability condition; discretization; Lie brackets; geometric phase; geometric numerical integration; local error computation.

### 1 Introduction

For nonlinear control systems, it is well known that nonintegrability conditions on the vector fields are at the basis of our notions of (nonlinear) controllability and observability [7, 25], as well as of many motion planning algorithms [16, 25]. If the system is driftless, then

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nonintegrability of the vector fields (and Lie bracket conditions) allows to produce motion in directions not spanned by the vector fields, as in the parallel parking of a car [14, 15, 24].

Such directions can be excited by generating cyclic input motion, the "macroscopic" equivalent of a Lie bracket. In some special cases the area of such cycles can be taken as indicator of the net displacement along these Lie bracket generated directions. This is true for instance when a notion of *geometric phase* can be associated to the system. This happens for instance when the state of the system has the structure of principal fiber bundle, and the state vector can be split into *shape* variables, directly controlled by external inputs, and *phase* variables depending from the shape variables, with a cyclic change in the former inducing a net displacement on the latter. In continuous-time, geometric phases have been extensively studied in different fields, like classical mechanics [17], quantum mechanics [4, 26], molecular systems, [18], robotics [16, 25] and control theory [7]. For such systems, periodic inputs can be used to induce non-periodic movements in the phase variables, see for instance [5, 9, 26] for applications to swimming bodies in fluids, [23] for the falling cat problem, and [14, 15, 24] for the already mentioned parallel parking of a car. In these cases, the amplitude of the phase displacement is proportional to the area of the cyclic path in shape space. In particular, a zero-area cycle yields no geometric phase.

For general driftless control systems, the equivalent of a "zero-area" input cycle is an input trajectory that goes forth and back to the same point, "rewinding itself" along the same path. In continuous-time, in correspondence of such a trivial cyclic motion also the state vector returns to its starting point, since no Lie bracket is excited when accomplishing it. We say in this case that we have an *involutive flow composition*. The concept can be extended from driftless control systems to time-reversible control systems [11].

The aim of this paper is to shown that the situation is different when the nonlinear system is discretized, in the sense that in discrete-time even an involutive flow composition may induce a net displacement in the state vector. We show that while an exact discretization obtained through a complete Taylor series expansion method does not violates involutivity, when instead the discrete-time system is obtained through an approximate discretization of a nonlinear system, for instance using an Euler method, then involutivity is violated. The displacement that is produced is related to the truncation error with respect to the exact discretization, and it appears to be both path-dependent and sampling length dependent. In particular, it tends to zero when the sampling interval tends to zero.

Although a large body of literature exists on determining discrete-time equivalents of the nonlinear notions used in control theory [3, 12, 19, 20, 22, 21], in our knowledge, the observation that involutivity of flow compositions is lost when a system is discretized appears to be novel. Even in the deeply investigated context of geometric phases, the properties of discrete time geometric phases have never been investigated, let alone the existence of phase motions induced by zero-area shape cycles.

In the context of numerical integration of ODEs, if a non-exact discretization of the system driven by an external input is integrated along a path and then rewinded along the same path, an estimate of the numerical integration error is then produced, estimate that can be used to improve the integration routine itself. For driftless bilinear control systems, the

numerical integration routine can be rendered exact. The observation that by decoupling involutive from non-involutive flows (i.e., those induced by Lie brackets) one can get an estimate on the fly of the local error of a numerical integration routine does not seem to appear in standard references on geometric numerical integration such as [6, 10] (although this author admits of not being an expert of the field).

A preliminary version of this work, focused mostly on the description of the zero-area discrete-time geometric phase, appears in the conference paper [2]. An application of the discrete-time geometric phase to financial models (namely, high frequency stock trading) appears in [1].

## 2 Involutivity of flows and discretization error for driftless nonlinear systems

#### 2.1 Involutivity of flows

Let f and g be  $C^{\infty}$  vector fields in  $\mathbb{R}^n$ . Their Lie bracket is the vector field

$$[f, g](x) = \frac{\partial g}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)g(x).$$

Denoting  $\Phi_t^f(x)$  the flow at time t along the vector field f at x (a local diffeomorphism), and by  $\gamma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$  the flow composition

$$\gamma(4t, x) = \Phi_t^{-g} \circ \Phi_t^{-f} \circ \Phi_t^g \circ \Phi_t^f(x) \tag{1}$$

then for t sufficiently small we have that

$$\gamma(4t, x) = x + t^2 \left(\frac{\partial g}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)g(x)\right) + \mathcal{O}(t^3)$$
(2)

i.e., the second order term of the expansion is given by the Lie bracket [f, g]. When the vector fields commute, i.e., when [f, g] = 0, then the right side of (2) is 0 and  $\gamma(4t, x) = x$  is the identity map.

When  $[f, g](x) = 0 \ \forall x \in \mathbb{R}^n$ , then a formula like (1) leads to  $\gamma(4t, x) = x$  not only locally (i.e., for small t) but also for arbitrarily large t. In the following we shall refer to this situation as *involutive flow composition*.

#### 2.2 Continuous-time involutive flows composition for driftless control systems

Consider the continuous-time nonlinear driftless system linear in the inputs:

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i.$$
(3)

For (3), as soon as time-varying input profiles  $u_i(t)$  are chosen, a flow composition like (1) is normally not involutive. The following proposition shows that for a special choice of input profiles, involutive flows can still be obtained. The input profile (eq. (4) below) is a trivial one: a "forth and back" trajectory along the same path.

**Proposition 1** Consider the system (3) and the input protocol

$$u_{i}(t) = \begin{cases} 0 & t < t_{1} \\ \alpha_{i} & t_{1} \leq t < t_{2} \\ 0 & t_{2} \leq t < t_{3} \\ -\alpha_{i} & t_{3} \leq t < t_{4} \\ 0 & t \geqslant t_{4} \end{cases}$$
(4)

where  $t_1$  and  $t_2$  are begin and end of an input step,  $t_3$  and  $t_4$  are begin and end of an identical input step of opposite sign (i.e.,  $t_2 - t_1 = t_4 - t_3$ ). We have the following:

**P1:** For the continuous-time system (3), the protocol (4) applied to all  $u_i$ , i = 1, ..., m, leads to x(t) = x(0) for  $t \ge t_4$ .

**Proof.** The flow of (3) under (4) is

$$\begin{aligned} x(t_2) &= \Phi_{t_2-t_1}^{\sum_{i=1}^{m} \alpha_i g_i} (x(0)) \\ x(t_4) &= \Phi_{(t_4-t_3)}^{-\sum_{i=1}^{m} \alpha_i g_i} (x(t_2)) = \Phi_{-(t_4-t_3)}^{\sum_{i=1}^{m} \alpha_i g_i} (x(t_2)). \end{aligned}$$

Since the vector field in the two flows is the same up to sign, the flows commute. The Lie bracket of identical vector fields is of course 0 and the Baker-Campbell-Hausdorff formula trivializes. Hence if  $t_4 - t_3 = t_2 - t_1$  we can write

$$x(t_4) = \Phi_{t_4-t_3}^{-\sum_{i=1}^m \alpha_i g_i} \circ \Phi_{t_2-t_1}^{\sum_{i=1}^m \alpha_i g_i} (x(0))$$
  
=  $\Phi_{-(t_4-t_3)+(t_2-t_1)}^{\sum_{i=1}^m \alpha_i g_i} (x(0)) = x(0)$ 

and **P1** is proven.

#### 2.3 Involutive flows and discretization

The Euler discretization of (3) is given by

$$x(k+1) = x(k) + h \sum_{i=1}^{m} g_i(x(k))u_i(k).$$
(5)

For inputs  $u_i$  that are piecewise-constant in each sampling interval h, an exact discretization of the system (3) is provided by a Taylor expansion [13]. It consists of the following infinite series

$$x(k+1) = x(k) + \sum_{j=1}^{\infty} B^{[j]}(x(k), u(k)) \frac{h^j}{j!}$$
(6)

where

$$B^{[1]}(x,u) = \sum_{i=1}^{m} g_i(x)u_i$$
  

$$B^{[j+1]}(x,u) = \frac{\partial B^{[j]}(x,u)}{\partial x} \sum_{i=1}^{m} g_i(x)u_i.$$
(7)

The truncation error of the Euler discretization is then

$$\epsilon(k) = \sum_{j=2}^{\infty} B^{[j]}(x(k), u(k)) \frac{h^j}{j!}.$$
(8)

From the expressions (5) and (6), it is clear that  $\lim_{h\to 0} \frac{x(k+h)-x(k)}{h} = \sum_{i=1}^{m} g_i(x(k))u_i(k)$  and  $\lim_{h\to 0} \frac{\epsilon(k)}{h} = 0.$ 

Let us look at the equivalent of Proposition 2 for the two discretizations (5) and (6).

**Proposition 2** Consider the Euler discretization (5) and the exact discretization (6) of the system (3). Consider the input protocols

$$u_{i}(k) = \begin{cases} 0 & k < k_{1} \\ \alpha_{i} & k_{1} \leqslant k < k_{2} \\ 0 & k_{2} \leqslant k < k_{3} \\ -\alpha_{i} & k_{3} \leqslant k < k_{4} \\ 0 & k \geqslant k_{4} \end{cases}$$
(9)

where  $k_1$  and  $k_2$  are begin and end of an input step,  $k_3$  and  $k_4$  are begin and end of an identical input step of opposite sign (i.e.,  $k_2 - k_1 = k_4 - k_3$ ). We have the following:

- **P2:** For the Euler discretization (5), the protocol (9) applied to all  $u_i$ , i = 1, ..., m, leads to  $x(k) \neq x(0)$  for  $k \ge k_4$ ;
- **P3:** For the exact discretization (6), the protocol (9) applied to all  $u_i$ , i = 1, ..., m, leads to x(k) = x(0) for  $k \ge k_4$ .

**Proof.** By construction, the exact discretization overlaps with the continuous-time solution at the sampling instants, meaning that **P3** must also hold. Hence the local error in the Euler discretization must corresponds to the cumulation of (8) along the path followed, meaning that generically **P2** will hold.

**Remark 1** The fact that all inputs are subject to an input protocol like (9) which leads to involutive flow composition is a key prerequisite for obtaining the properties **P1** and **P3**. In fact, as soon as for a flow composition involutivity is lost and non-zero Lie brackets arise, the Baker-Campbell-Hausdorff formula is no longer trivial and  $x(t_4) \neq x(0)$  in general (and similarly for the exact Taylor discretization).

## 2.4 Extension to general input protocols and to time-reversible control systems

For the driftless control system (3), involutive flow compositions can be obtained from more general input paths than the piecewise constant profile (4). For instance

$$u_{i}(t) = \begin{cases} \alpha_{i}(t) & 0 \leq t < \tau \\ -\alpha_{i}(2\tau - t) & \tau \leq t < 2\tau \\ 0 & t \geq 2\tau \end{cases}$$
(10)

where  $\alpha_i(t)$  is any time-varying function, is such that it produces an involutive flow and  $x(2\tau) = x(0)$ . In fact, for  $t \in [\tau, 2\tau]$ ,  $\Phi^{\sum_{i=1}^{m} g_i(x)u_i}$  rewinds along the same trajectory followed for  $t \in [0, \tau]$ . More specifically, if  $s = t - \tau$  and  $\alpha_i([s_1, s_2])$  denotes the value assumed by  $\alpha_i$  in the interval  $[s_1, s_2]$ , for  $t \in [\tau, 2\tau]$  we have

$$x(t) = \underbrace{\Phi_{s}^{-\sum_{i=1}^{m} g_{i}(x)\alpha_{i}([\tau, \tau-s])} \circ \Phi_{s}^{\sum_{i=1}^{m} g_{i}(x)\alpha_{i}([\tau-s, \tau])}}_{\mathrm{Id}} \circ \Phi_{\tau-s}^{\sum_{i=1}^{m} g_{i}(x)\alpha_{i}([0, \tau])} (x(0))$$

$$= \Phi_{\tau-s}^{\sum_{i=1}^{m} g_{i}(x)\alpha_{i}([0, \tau])} (x(0))$$
(11)

i.e., the second part of (10) indeed cancels the first part, until  $x(2\tau) = x(0)$ .

The driftless control system (3) is a special case of the larger class of so-called timereversible control systems to which the result of Proposition 1 and 2 can be extended. A system  $\dot{x} = f(x, u), u \in U \subset \mathbb{R}^m$  is called *time-reversible* [11] if  $\exists$  a function  $v(u) : U \to U$ and a scalar positive function  $\lambda$  such that

$$f(x, u) = -\lambda(x, u)f(x, v(u)) \qquad \forall (x, u) \in \mathbb{R}^n \times U.$$
(12)

For the input protocol (10), it is enough to choose  $u_i(t) = \alpha_i(t)$  for  $t \in [0, \tau]$  and then  $v_i(u)$  and  $\lambda(x, u)$  so that (12) holds for  $t \in [\tau, 2\tau]$  to obtain an involutive flow composition.

In particular, for the driftless system (3) it is  $\lambda = 1$  and v(u) = -u, provided that U is symmetric with respect to the origin of  $\mathbb{R}^m$ . Then, from (12),  $\left(\Phi_t^{\sum_{i=1}^m g_i u_i}\right)^{-1}(x) = \Phi_t^{-\sum_{i=1}^m g_i u_i}(x)$ , which leads to (11).

# 3 Truncation error of involutive flows and geometric phase

For a special class of 2-input driftless systems in dimension 3, the situation we are investigating has a particularly significant geometric meaning. Let us consider a well-known example, the Brockett nonholonomic integrator [8].

#### 3.1 Continuous-time Brockett integrator

In continuous-time, the Brockett integrator is the driftless control system

$$\dot{x}_1 = u_1$$
  
 $\dot{x}_2 = u_2$   
 $\dot{x}_3 = x_1 u_2.$ 
(13)

In the following  $x_1$  and  $x_2$  will be denoted *shape* variables, and  $x_3$  as *phase* variable. Assume that the control inputs  $u_1$  and  $u_2$  correspond to the piecewise constant trajectories shown in the top plot of panel (a) of Fig. 1. Then a cyclic motion is produced in shape space S which results in a net displacement in the phase variable  $x_3$ , meaning that the flow composition is not involutive, see panel (b) of Fig. 1. The geometric interpretation of this result is that if we consider  $x \in \mathcal{M} \subset \mathbb{R}^3$  and the projection to the shape space

$$\pi : \mathcal{M} \to \mathcal{S} \subset \mathbb{R}^2$$
$$x \mapsto x_s = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

then, given x(0), for each trajectory  $\gamma : [0,t] \to S \exists$  a unique  $x(t) \in \mathcal{M}$  such that for the solution of (13),  $x(t) = \pi^{-1}(\pi(x(t)))$ , i.e., the geometric phase variable  $x_3(t)$  corresponding to  $\gamma(t)$  is unique. In particular, if  $\Gamma : [0,T] \to S$  is a closed shape curve enclosing an area  $\Omega$ , then the geometric phase (or "holonomy", [7]) of  $\Gamma$  is

$$x_3(T) = x_3(0) + \oint_{\Gamma} x_1 dx_2$$

or, by Stokes theorem,

$$x_{3}(T) = x_{3}(0) + \int_{\Omega} d(x_{1}dx_{2})$$
  
=  $x_{3}(0) + \int_{\Omega} dx_{1}dx_{2} = x_{3}(0) + \omega$  (14)

where  $\omega$  is the area of  $\Omega$ .

When instead the input trajectories  $u_1$  and  $u_2$  are identical and equal to (4), so are those of the shape variables  $x_1$  and  $x_2$ , meaning that a cyclic trajectory of area zero is produced in shape space, see panels (c) and (d) of Fig. 1. From (14), in this case the phase variable  $x_3$  shows no net displacement at the end of the cycle, see also below for an alternative calculation. In other words, the flow composition is involutive and **P1** holds.

#### **3.2** Discrete-time Brockett integrator

Let us now consider a discretized version of the system (13), for instance obtained replacing the derivative operator with an Euler difference, with sampling time  $h = \Delta t$  assumed



Figure 1: Effect of cyclic shape trajectories in the continuous-time Brockett integrator (13). Left column: time profiles of the variables. Right column: corresponding shape (orange) and phase space (blue) profiles. The starting point is given in green and the end point in red. In the top row the area of the shape cycle in  $(x_1, x_2)$  is nonzero, and so is the phase  $(x_3)$  displacement. In the bottom row the shape cycle has zero area and so does the phase displacement (involutive flow composition).

constant:

$$x(k+1) = x(k) + h \begin{bmatrix} 1\\0\\0 \end{bmatrix} u_1(k) + h \begin{bmatrix} 0\\1\\x_1(k) \end{bmatrix} u_2(k).$$
(15)

For this system, the shape and phase variables corresponding to the same input patterns as in Fig. 1 are shown in Fig. 2. While the case of non-zero area shape cycle is similar (panels (a) and (b)), the case of zero-area shape cycle (panels (c) and (d)) is not. In particular a non-zero net displacement in the phase variable  $x_3$  happens also when the area of the space cyclic trajectory is zero, see panels (c) and (d) of Fig. 2, i.e., **P2** holds. Let us compute explicitly the geometric phase accumulated along the path. Denote  $\ell = k_2 - k_1 = k_4 - k_3$  the step length in number of samples,  $\ell \ge 1$  ( $\ell = 5$  in Fig. 2). In correspondence of the input pattern (9) with  $\alpha_1 = \alpha_2 = \alpha$ , the solution of (15) is



Figure 2: Effect of cyclic shape trajectories in the discrete-time system (15). Same quantities and color code of Fig. 1 are shown. In the top row the area of the shape cycle in  $(x_1, x_2)$  is nonzero, and so is the phase  $(x_3)$  displacement. In the bottom row the shape cycle has zero area but the phase displacement is still nonzero (violation of involutivity).

- for  $k \leq k_1$ x(k) = x(0) (16)
- for  $k_1 < k \leq k_2$  $x(k) = x(0) + (k - k_1) \begin{bmatrix} 1\\ 1\\ x_2(0) \end{bmatrix} h\alpha + \begin{bmatrix} 0\\ 0\\ \sum_{i=1}^{k-k_1-1} i \end{bmatrix} h^2 \alpha^2$ (17)

$$x(k) = x(0) + \ell \begin{bmatrix} 1\\1\\x_2(0) \end{bmatrix} h\alpha + \begin{bmatrix} 0\\0\\\sum_{i=1}^{\ell-1}i \end{bmatrix} h^2 \alpha^2$$
(18)

• for  $k_3 < k \leq k_4$ 

• for  $k_2 < k \leq k_3$ 

$$x(k) = x(0) + (k_4 - k) \begin{bmatrix} 1\\ 1\\ x_2(0) \end{bmatrix} h\alpha + \begin{bmatrix} 0\\ 0\\ \sum_{i=1}^{k_4 - k} i - \ell \end{bmatrix} h^2 \alpha^2$$
(19)

• for  $k > k_4$ 

$$x(k) = x(0) - \begin{bmatrix} 0\\0\\\ell \end{bmatrix} h^2 \alpha^2.$$
(20)

Since  $x_1(k) = x_2(k) \ \forall k$ , the shape trajectory has indeed zero area. Furthermore, since for  $k > k_4 \ x_i(k) = x_i(0)$ , i = 1, 2, the shape trajectory is also periodic. However, for  $k > k_4$ ,  $x_3(k) \neq x_3(0)$ , as a geometric phase proportional to  $\ell h^2 \alpha^2$  has been generated. Therefore it follows that an area rule like (14) cannot hold for discrete-time systems.

To see why such a geometric phase appears in this case, let us look at the summations in (16)-(20): during the positive input step  $(k_1 \leq k < k_2)$ , the  $x_3$  variable builds up the partial sum  $\sum_{i=1}^{\ell-1} i$  (starting from 0, then 1, until  $\ell-1$ ). However, when entering the negative input step  $(k_3 \leq k < k_4)$ , the first term subtracted to that summation is  $\ell$  (then  $\ell-1, \ell-2, \ldots, 1$ , until complete erasure of the summation). It is this negative term  $\ell$  which leads to a nonzero phase motion.

If the continuous-time input steps of (4) have amplitude 1 and area 1, i.e.,  $\int_{t_1}^{t_2} u_i(\tau) d\tau = 1$ where  $t_i = k_i h$ , then for the discrete-time system, with  $\alpha_i = 1$ ,

$$\sum_{k=k_1}^{k_2} hu_i(k) = h(k_2 - k_1) = h\ell.$$

Hence when the sampling time  $h \to 0$ , the constraint  $h\ell = 1$  becomes  $\lim_{h\to 0} h\ell = 1$  i.e.,  $\lim_{h\to 0} \ell = \infty$  but  $\lim_{h\to 0} \ell h^2 = 0$ , meaning that in (20) the geometric phase disappears when the Euler difference converges to a continuous-time differential operator.

Apart from the sampling time, this zero-area geometric phase appears to be dependent also on the path followed. For instance, if we apply sinusoidal inputs  $(u_1 = u_2)$  to the system (15) then we obtain the phase shown in Fig. 3. When the sampling time is divided by two, the geometric phase accumulated along the sinusoidal path decreases. Also reducing the input amplitude (but not h) reduced the geometric phase.

#### 3.3 Geometric phase and exact discretization for the Brockett integrator

If we use homogeneous coordinates  $\bar{x} = \begin{bmatrix} x \\ -1 \end{bmatrix}$ , then (13) can be rewritten as the bilinear system

$$\dot{\bar{x}} = (A_1 u_1 + A_2 u_2)\bar{x} \tag{21}$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & | & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 1 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & | & 0 \end{bmatrix}.$$



Figure 3: Sinusoidal inputs for the discrete-time Brockett integrator (15). In blue the input has amplitude 1 and sampling time h = 0.2 (the period is T = 10). In red the sampling time is h = 0.1. In violet the amplitude of the input is reduced to 0.5 (and h = 0.2). In green the integral curve of the continuous time system (13) is shown. The inset in the lower panel shows the end-point of the curve (i.e., the geometric phase accumulated by  $x_3$  along a zero-area cycle).

When the piecewise constant input pattern (4) is applied to (21), the explicit solution one gets for  $\alpha_i = \alpha$ , i = 1, 2, is

$$\bar{x}(t_2) = e^{(A_1 + A_2)(t_2 - t_1)\alpha} \bar{x}(0)$$
(22)

and

$$\bar{x}(t_4) = e^{-(A_1 + A_2)(t_4 - t_3)\alpha} \bar{x}(t_2).$$
(23)

If  $t_2 - t_1 = t_4 - t_3$ , then

 $\bar{x}(t_4) = \bar{x}(0),$ 

as the two exponentials in (22) and (23) are one the inverse of the other. Hence indeed the geometric phase of (13) is zero for an input trajectory like (4).

In discrete-time, using homogeneous coordinates the system (15) becomes

$$\bar{x}(k+1) = (I + h(A_1u_1 + A_2u_2))\,\bar{x}(k).$$
(24)

For the input pattern (9) with  $\alpha_i = \alpha$ , this leads, at the end of the positive step to

$$\bar{x}(k_2) = (I + h(A_1 + A_2)\alpha)^{\ell} \bar{x}(0),$$

and at the end of the negative step, to

$$\bar{x}(k_4) = (I - h(A_1 + A_2)\alpha)^{\ell} \bar{x}(k_2)$$

(explicit expressions coincide obviously with (16)-(20)). However, the matrix  $(I - (A_1 + A_2)\alpha)^{\ell}$ 

is not the inverse of  $(I + (A_1 + A_2)\alpha)^{\ell}$ . In particular

$$(I - h(A_1 + A_2)\alpha)^{\ell} (I + h(A_1 + A_2)\alpha)^{\ell} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -\ell h^2 \alpha^2\\ \hline 0 & 0 & 0 & 0 \end{bmatrix}$$

meaning that the expression (20) is obtained for  $x(k_4)$  at the end of the cycle.

For the Brockett integrator in the homogeneous coordinates just mentioned it is quite easy to compute the exact discretization. In fact, the matrices  $A_1$  and  $A_2$  are nilpotent:  $A_i^2 = 0$ , i = 1, 2, and the only nonzero matrix product is

Hence we have the exact series expansion

$$e^{\tau(A_1u_1+A_2u_2)} = I + \tau(A_1u_1 + A_2u_2) + \frac{\tau}{2!}(A_1u_1 + A_2u_2)^2$$
  
=  $I + \tau(A_1u_1 + A_2u_2) + \frac{\tau}{2!}A_2A_1u_1u_2.$ 

Calling  $h = \tau$ , the system (13) admits the exact discretization given by the (complete) Taylor expansion

$$x(k+1) = x(k) + h \begin{bmatrix} 1\\0\\0 \end{bmatrix} u_1(k) + h \begin{bmatrix} 0\\1\\x_1(k) \end{bmatrix} u_2(k) + \frac{h^2}{2} \begin{bmatrix} 0\\0\\1 \end{bmatrix} u_1(k) u_2(k).$$
(25)

The system (25) has no geometric phase when the input protocol (9) is applied, i.e., **P3** holds. It is clear from (20) that the truncation error, i.e., the extra term distinguishing (25) from (15) is exactly equal to half the geometric phase at the end of a cycle induced by (9).

**Remark 2** It is worth observing that when the input pattern of panels (a) and (b) of Fig. 2 is used, the Euler discretization (15) produces an exact result at the end of the cycle. In

fact,

$$u_{1}(k) = \begin{cases} 0 & k < k_{1} \\ \alpha & k_{1} \leqslant k < k_{2} \\ 0 & k_{2} \leqslant k < k_{3} \\ -\alpha & k_{3} \leqslant k < k_{4} \\ 0 & k_{4} \leqslant k < k_{5} \\ 0 & k \geqslant k_{5}. \end{cases} \quad u_{2}(k) = \begin{cases} 0 & k < k_{1} \\ 0 & k_{1} \leqslant k < k_{2} \\ \alpha & k_{2} \leqslant k < k_{3} \\ \alpha & k_{2} \leqslant k < k_{3} \\ 0 & k_{3} \leqslant k < k_{4} \\ -\alpha & k_{4} \leqslant k < k_{5} \\ 0 & k \geqslant k_{5} \end{cases}$$

implies that  $u_1(k)u_2(k) = 0 \ \forall k$ , hence the extra term present in (25) always gives zero contribution.

#### **3.4** Other examples

While the results of Proposition 1 and 2 are valid for general driftless (or time-reversible) systems, even in dimension 3 it is not always possible to split the state into shape and phase variables, as the following example shows.

**Example 2** Consider the kinetic wheel rolling without slipping of Fig. 4, whose equations



Figure 4: Example 2: unicycle kinetics.

are

$$\dot{x}_1 = u_1 \cos(x_3)$$
  
 $\dot{x}_2 = u_1 \sin(x_3)$  (26)  
 $\dot{x}_3 = u_2$ 

and apply the input protocol (4). As expected, (26) obeys **P1**, while for the Euler discretization the truncation error is shared among the variables  $x_1$  and  $x_2$ . Hence we cannot talk about zero-area shape path and scalar phase variable.

For systems in  $\mathbb{R}^3$ , only when we have two independently controlled integrators then we can talk about zero-area geometric phase. In that case, the vectors fields need not be nilpotent, as the following example shows.



Figure 5: Example 2 (unicycle kinetics): simulations. (a): inputs. (b) state variables. (c): 3D trajectories.

Example 3 The Euler discretization of the non-nilpotent system

$$\dot{x} = \begin{bmatrix} 1\\0\\\sin(x_2) \end{bmatrix} u_1 + \begin{bmatrix} 0\\1\\\cos(x_1) \end{bmatrix} u_2.$$
(27)

is given by

$$x(k+1) = x(k) + h \begin{bmatrix} 1\\0\\\sin(x_2(k)) \end{bmatrix} u_1(k) + h \begin{bmatrix} 0\\1\\\cos(x_1(k)) \end{bmatrix} u_2(k)$$
(28)

Computing the terms (7) explicitly, the exact Taylor discretization (6) is (the index k in the right hand side is omitted to save some space):

$$\begin{aligned} x(k+1) &= x(k) + h \begin{bmatrix} 1\\0\\\sin(x_2) \end{bmatrix} u_1 + h \begin{bmatrix} 0\\1\\\cos(x_1) \end{bmatrix} u_2 \\ &+ \begin{bmatrix} 0\\\frac{\cos(x_1)}{u_1^2} (\sin(u_1h) - u_1h) + \frac{\sin(x_1)}{u_1^2} (\cos(u_1h) - 1) \end{bmatrix} u_1 u_2 \\ &+ \begin{bmatrix} 0\\\frac{\sin(x_2)}{u_2^2} (\sin(u_2h) - u_2h) - \frac{\cos(x_1)}{u_2^2} (\cos(u_2h) - 1) \end{bmatrix} u_1 u_2. \end{aligned}$$
(29)

The integral curves of the systems (28) and (29) are compared in Fig. 6 on the same zero-area shape input trajectory (9). While (28) shows a geometric phase, (29) matches exactly (27) at all sampling instants.

Another class of systems to which the idea of geometric phase (in vectorial form) can be extended straightforwardly is given by driftless systems in  $\mathbb{R}^n$  having two independently controlled integrators plus n-2 nonlinear terms.



Figure 6: Comparing Euler and exact discretization of Example 3. The Euler discretization (28) (in blue) shows a geometric phase. The exact discretization (29) (in red) shows no geometric phase and overlaps exactly with the true continuous-time system (green).

**Example 4** A classical example is given by systems in chained form [24]. These *n*-dimensional driftless systems are of the form

$$\dot{x}_1 = u_1 
\dot{x}_2 = u_2 
\dot{x}_3 = x_2 u_1 
\vdots 
\dot{x}_n = x_{n-1} u_1$$
(30)

and can be readily expressed as a bilinear system in homogeneous coordinates as in (21) with matrices

Since  $A_1$  and  $A_2$  are still nilpotent, identical arguments to those carried out in the Section 3.3 still hold. Clearly, in this case, the truncation error is in the variables  $x_3 \div x_n$ , hence the geometric phase is vectorial.

## 4 Involutive correction terms for numerical integration algorithms

The idea of using input protocols that induce involutive flow compositions on a continuoustime system is key for decoupling the discretization effects from the nonintegrability effects of more general flows. This possibility can be employed in a numerical scheme that aims to compensate for the truncation errors induced by a numerical integration. The idea is explained below for the Euler discretization of the driftless system (3) and illustrated in Fig. 7 for the special case of a system in  $\mathbb{R}^3$  with a shape/phase decomposition (as in the Brockett integrator).

#### Algorithm 1 (Correction to Euler discretization.)

Step 1: Euler discretization step.

- input:  $u_i(k) = \text{const}$
- preliminary state update:

$$x^{\text{tmp}}(k+1) = x(k) + h \sum_{i=1}^{m} g_i(x(k))u_i(k).$$

- Step 2: Introduce an extra time step with reversed input and compute the local error on the "cycle".
  - input:  $-u_i(k)$
  - extra state update:

$$x^{\text{ext}}(k+1) = x^{\text{tmp}}(k+1) - h \sum_{i=1}^{m} g_i(x^{\text{tmp}}(k+1))u_i(k).$$

• local error:

$$e(k+1) = x^{\text{ext}}(k+1) - x(k)$$
  
=  $h \sum_{i=1}^{m} g_i(x(k))u_i(k) - h \sum_{i=1}^{m} g_i(x^{\text{tmp}}(k+1))u_i(k).$  (31)

Step 3: Correct the preliminary state update using the local error

• final state update:

$$x^{\text{corr}}(k+1) = x^{\text{tmp}}(k+1) - \frac{e(k+1)}{2}$$
  
=  $x(k) + h \frac{\sum_{i=1}^{m} g_i(x(k))u_i(k)}{2} + h \frac{\sum_{i=1}^{m} g_i(x^{\text{tmp}}(k+1))u_i(k)}{2}.$  (32)

The correction one obtains in this way is similar, but not identical, to the trapezoidal rule of an Euler method. The difference is that there is no need to solve an implicit algebraic equation, as the method is completely explicit, see (32). The computational burden of the method corresponds to the calculations made in the extra time step, which roughly double the total computational cost.



Figure 7: Computing a correction to a numerical integration routine (Algorithm 1).

**Example** (Brockett integrator) The result is shown in Fig. 8. In this case we known that the geometric phase is twice the error of a piecewise constant "forth and back" input trajectory, and corresponds exactly to the truncation error of an Euler discretization. Hence the correction proposed with Algorithm 1 "exactifies" the Euler method in this case.



Figure 8: Numerical correction with Algorithm 1 for the Brockett integrator. (a): input; (b): cumulated local error e; (c): true (continuous-time) trajectory and corrected Euler discretization  $x^{\text{corr}}$  (blue, overlapping), and Euler discretization  $x^{\text{tmp}}$  (red).

**Remark 3** The numerical correction given by Algorithm 1 exactifies the Euler discretization whenever the system can be written as a driftless bilinear system. In fact, for it the formal exponentials describing the flow are matrix exponentials, hence the local error (31) is exactly twice the truncation error of the Euler discretization (as shown in Section 3.3 for the Brockett integrator).

**Example 5** For the following  $\mathbb{R}^3$  example

$$\dot{x}_1 = u_1$$
  
 $\dot{x}_2 = u_2$   
 $\dot{x}_3 = \tanh(x_1)x_2^2u_2 - x_1^3u_1$ 

the correction computed via Algorithm 1 is not exact, but still can improve the Euler discretization, see Fig. 9 (where we take as "truth" for the continuous-time trajectory the result of Matlab's ode45 numerical integration with very low absolute and relative tolerances).



Figure 9: Numerical correction to an Euler discretization made with Algorithm 1 for Example 5.

## 5 Conclusion

In this paper we have shown through examples and explicit calculations that discrete-time driftless nonlinear dynamical systems which are not exact Taylor expansions of continuous-time systems violate the flow involutivity conditions which are instead obeyed by their continuous-time counterpart: even if a cyclic motion does not excite any Lie bracket (i.e., it is involutive) the resulting discrete-time trajectories are not periodic in some of the variables. For the special case of systems admitting a shape/phase decomposition, our results correspond to violation of the so-called area rule for the geometric phase produced by cyclic motions in shape space. It is also shown how such property can in principle be exploited to compute local errors of numerical integration routines.

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