Tentamen 2017
Robust Multivariable Control
Duration 3 days (3×24 hrs)

Anders Helmersson
`http://www.control.isy.liu.se/~andersh`
email: `andershelmersson@liu.se`
phone: 013-157 888 (home)
013-482 8419 (RUAG Space)
0734 278 419 (mobile)

1

A system has a time delay of 0.1 *seconds and a pole in* 1 *rad/s.*

What is the lowest possible peaks of the sensitivity functions S and T in closed-loop.

The lower bound on $S = 1/(1 + KG)$ follows from

$$\int_0^\infty \log |S(j\omega) d\omega = \pi,$$

from which we obtain $\log |S(j\omega)| > 0$ as a strict bound. However, the delay limits the bandwidth to about 10 rad/s, which means that $\log |S(j\omega)| > \pi/10$ 0.3 Thus $|S| > 1.3$ (about 2.6 dB).

The bound of T can be obtained from Skogestad (5.19) on $G(s) = \frac{e^{-0.1s}}{s-1}$.

$$\|T\|_\infty \geq \prod_j \frac{|\bar{z}_j + p|}{|z_j - p|} e^{0.1\,p}$$

The bound of S can also be computed using this expression. Thus, the peaks of S and T are at least 1.1052.

2

Let $\Delta = \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_1 \end{bmatrix}$. *Determine the set of matrices, D, that commutes with Δ: $D\Delta = \Delta D$.*

The set of matrices that commute with Δ is given by $D = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_1 \end{bmatrix}$.

Consider the problem of minimizing $\bar{\sigma}(DMD^{-1})$ with respect to all non-singular such D. Can you be sure that you always find the global minimum?

This problem can be rewritten as $M^T PM \prec \gamma^2 P$, where

$$P = \begin{bmatrix} d_1^2 + d_2^2 & 2d_1 d_2 \\ 2d_1 d_2 & d_1^2 + d_2^2 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_1 \end{bmatrix} \succ 0.$$

Note that the positive definiteness of P can be expressed as $p_1 \pm p_2 > 0$.

From any such P we can find a (not necessarily unique) matrix D of the desired form.

Can this structure be extended to higher dimensions?

For higher dimensions

$$\Delta = \begin{bmatrix} \delta_1 & \delta_2 & \cdots & \delta_n \\ \delta_n & \delta_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_2 \\ \delta_2 & \cdots & \delta_n & \delta_1 \end{bmatrix}$$

which is a so called circulant matrix (a special case of Toeplitz). The commuting D and $P = D^T D$ have the same structure. Here D can be computed by Discrete Fourier Transform (DFT), in Matlab `d=ifft(sqrt(fft(p)))`.

3

Minimize the maximum singular value of

$$\begin{bmatrix} 1 & x \\ 3 & 4 \end{bmatrix}$$

with respect to x. Rewrite $\bar{\sigma}\left(\begin{bmatrix} 1 & x \\ 3 & 4 \end{bmatrix}\right) \leq \gamma$ as

$$\left[\begin{array}{cc|cc} -\gamma & 0 & 1 & x \\ 0 & -\gamma & 3 & 4 \\ \hline 1 & 3 & -\gamma & 0 \\ x & 4 & 0 & -\gamma \end{array}\right] \preceq 0$$

and apply the elimination lemma on x. This gives

$$\begin{bmatrix} -\gamma & 3 & 4 \\ 3 & -\gamma & 0 \\ 4 & 0 & -\gamma \end{bmatrix} \preceq 0, \Leftrightarrow \gamma \geq \bar{\sigma}\left(\begin{bmatrix} 3 & 4 \end{bmatrix}\right) = 5$$

and

$$\begin{bmatrix} -\gamma & 0 & 1 \\ 0 & -\gamma & 3 \\ 1 & 3 & -\gamma \end{bmatrix} \preceq 0 \Leftrightarrow \gamma \geq \bar{\sigma}\left(\begin{bmatrix} 1 & 3 \end{bmatrix}\right) = \sqrt{10}$$

Thus, $\gamma \geq 5$.

Next, minimize

$$\begin{bmatrix} 1 & x & y \\ 3 & 4 & z \\ 2 & 5 & 6 \end{bmatrix}$$

with respect to x, y and z.

As before, apply the elimination lemma first on $\begin{bmatrix} x & y \end{bmatrix}$ which gives

$$\gamma \geq \bar{\sigma}\left(\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right) = \sqrt{14}$$

and

$$\gamma \geq \bar{\sigma}\left(\begin{bmatrix} 3 & 4 & z \\ 2 & 5 & 6 \end{bmatrix}\right)$$

Next, apply the elimination lemma on the last inequality:

$$\gamma \geq \bar{\sigma}\left(\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}\right) \approx 7.2854$$

and

$$\gamma \geq \bar{\sigma}\left(\begin{bmatrix} 2 & 5 & 6 \end{bmatrix}\right) = \sqrt{65}$$

Thus, $\gamma \geq \sqrt{65} \approx 8.0623$.

Are the optimal values of x, y and z unique (in both examples)? In the first problem

$$\left[\begin{array}{c|cc|c} -\gamma & 0 & 1 & x \\ \hline 0 & -\gamma & 3 & 4 \\ 1 & 3 & -\gamma & 0 \\ \hline x & 4 & 0 & -\gamma \end{array}\right] \preceq 0$$

perform a Schur complement with $\gamma = 5$:

$$\begin{aligned} &\begin{bmatrix} -5 & x \\ x & -5 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{75}{16} & x + \frac{3}{4} \\ x + \frac{3}{4} & 0 \end{bmatrix} \preceq 0 \end{aligned}$$

Here we have the unique solution $x = -\frac{3}{4}$ since the off-diagonal element must be 0.

Similar for the second problem. First, we can find a unique $z = -13/3$ to

$$\bar{\sigma}\left(\begin{bmatrix} 3 & 4 & z \\ 2 & 5 & 6 \end{bmatrix}\right) = \sqrt{65}$$

Next,

$$\left[\begin{array}{c|ccc|cc} -\gamma & 0 & 0 & 1 & x & y \\ \hline 0 & -\gamma & 0 & 3 & 4 & -13/3 \\ 0 & 0 & -\gamma & 2 & 5 & 6 \\ 1 & 3 & 2 & -\gamma & 0 & 0 \\ \hline x & 4 & 5 & 0 & -\gamma & 0 \\ y & -13/3 & 6 & 0 & 0 & -\gamma \end{array}\right] \preceq 0$$

with Schur complement with $\gamma = \sqrt{77}$:

$$\begin{bmatrix} -\gamma & x & y \\ x & -\gamma & 0 \\ y & 0 & -\gamma \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 4 & 5 & 0 \\ -13/3 & 6 & 0 \end{bmatrix} \begin{bmatrix} -\gamma & 0 & 3 \\ 0 & -\gamma & 2 \\ 3 & 2 & -\gamma \end{bmatrix}^{-1} \begin{bmatrix} 0 & 4 & -13/3 \\ 0 & 5 & 6 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|cc} -8.6379 & x + 0.3437 & y - 0.0156 \\ \hline x + 0.3437 & -3.2408 & 1.4043 \\ y - 0.0156 & 1.4043 & -2.5307 \end{array}\right] \preceq 0$$

The solution $\begin{bmatrix} x & y \end{bmatrix}$ is not unique since the diagonal blocks are non-singular. Possible solutions are

$$\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} -0.3437 & 0.0156 \end{bmatrix} + \delta \begin{bmatrix} 3.2408 & -1.4043 \\ -1.4043 & 2.5307 \end{bmatrix}^{\frac{1}{2}}$$

for $\|\delta\| \leq \sqrt{8.6379}$.

4

The Riccati equation can be solved by finding the eigenvalues and eigenvectors of the Hamiltonian matrix,

$$H = \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}, \quad \text{where} \quad R = R^T,\ Q = Q^T.$$

Show that the eigenvalues are symmetric with respect to the real and imaginary axes.

If one eigenvector is known, how can you compute the related eigenvectors from it?

In order to find a real solution X to the Riccati equation

$$XA + A^T X + Q + XRX = 0,$$

how should you combine the eigenvectors?

See text book solutions

5

Consider the three first-order systems with inputs $\begin{bmatrix} w \\ u \end{bmatrix}$ and outputs $\begin{bmatrix} z \\ y \end{bmatrix}$:

(1)

$$G_1(s) = \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right]$$

(2)

$$G_2(s) = \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 1 & 1 \\ 1 & 0 & 0 \end{array}\right]$$

(3)

$$G_3(s) = \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

Consider the H_∞ controller problem, using y as input and u as output, for these three systems.

What is the smallest achievable gains and the corresponding controllers for these systems?

Can you find a zeroth order controller for each case?

Are the controllers acceptable in all cases?

G_1

$$G_1(s) = \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right]$$

This is a regular problem, which satisfies the rank conditions on D_{12} and D_{21}. The optimal $\gamma^* = 0$ with $K = -1$. Here `hinfsyn` gives

$$K = \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & -1 \end{array}\right] = -1,$$

which can be reduced to a static gain of -1.

G_2

$$G_2(s) = \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 1 & 1 \\ 1 & 0 & 0 \end{array}\right]$$

For this problem $D_{21} = 0$. The optimal $\gamma^* = 1$. Here `hinfsyn` produces

$$K = \left[\begin{array}{c|c} -a & a \\ \hline -1 & 0 \end{array}\right] = -\frac{a}{s+a},$$

where a approaches infinity as γ approaches 1. The controller approaches a static gain $K = -1$. Actually, any static controller $K \leq -1$, achieves the minimum gain.

G_3

$$G_3(s) = \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

For this problem both $D_{12} = D_{21} = 0$. The optimal $\gamma^* = 0$. Here `hinfsyn` produces

$$K = \left[\begin{array}{c|c} -2a & a \\ \hline -a & 0 \end{array}\right] = -\frac{a^2}{s+2a},$$

where a approaches infinity as γ approaches 0, which means that the gain of K increases without bound. A static controller with gain $-k$ works and produces a $\gamma = 1/k$.

In this problem the optimal control problem produces infinite gain, which is not acceptable. Here the problem lacks a bound on the gain of K.

The first problem produces controllers that is acceptable. The second problem has solutions that have no bound on the gain. The third problem has an infinite solution for obtaining the optimal gain. Both problem 2 and 3 need to be respecified.

6

Robin is making a toy robot vehicle consisting of a 0.6-meter vertical rod, on to which the lower end two small electrical motors are attached. Each motor drives a wheel with a diameter of 100 mm. Also, an encoder is attached measuring the rotation of the motor shaft. The motors are controlled by a microprocessor, which uses the encoder signals and gyro data from a sensor mounted on the rod.

Use the following model for the pitch dynamics: θ is the angle of the rod relative to the vertical, ϕ is the mean rotation angle of the motor shafts relative to the robot measured by the encoder.

$x = (\theta + \phi)r$ (location of the center of the wheel), $r = 0.05\ m$.

$$\begin{bmatrix} I + m\ell^2 & m\ell\cos\theta \\ m\ell\cos\theta & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} gm\ell\sin\theta \\ 0 \end{bmatrix} + m\ell\dot{\theta}\sin\theta \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 1 \\ -1/r \end{bmatrix} u$$

where $I = 0.03$ kgm^2, $m = 0.5$ kg, $\ell = 0.2$ m and $g = 9.81$ m/s^2. The control signal, u, is the total motor torque produced by the two motors. The same command is given to both motors.

We have neglected the slipping of the wheels relative to the ground surface.

A linearized model around $\theta = 0$ and $\dot{\theta} = 0$ becomes

$$\begin{bmatrix} 0.05 & 0.1 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0.981 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ -20 \end{bmatrix} u$$

or

$$\frac{d}{dt} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \theta \\ x \end{bmatrix} = \begin{bmatrix} 0 & 0 & 32.7 & 0 \\ 0 & 0 & -6.54 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \theta \\ x \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 500 \\ -220 \\ 0 \\ 0 \end{bmatrix} u$$

and

$$\begin{bmatrix} \dot{\theta} \\ \phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 20 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \theta \\ x \end{bmatrix}$$

The measurements ϕ from the encoder and $\dot{\theta}$ from the gyro are available for the controller to produce the torque commands, u, to the motors.

Consider the problem of designing a controller with the following requirements:

- The controller should stabilize the vehicle;
- The controller shall be able to hande a delay of 0.02 s in the loop;
- The gain and phase margins should be adequate, aim at 5 dB and 30 deg at the input of the motor;
- The controller shall be able to accept a reference input in x;
- The attitude angle, θ, shall be zero in steady state;
- Try to make the step response in x as fast as possible and with resonable overshoot.

Loop shaping

We start by trying to use a simple loop shaping technique with $W_1 = 1$ and $W_2 = \begin{bmatrix} w_{21} & 0 \\ 0 & w_{22} \end{bmatrix}$. Before doing this we add a delay in the loop using a Padé first-order approximation.

To arrive at this we start by only using the $\dot{\theta}$ output and finding a loop shaping controller where w_{22} is very small and w_{21} is adjusted to obtain a low value of γ. Here we can choose $w_{21} = 0.15$. We then gradually increase w_{22} until γ get close to 3-3.5. This results in $w_{22} = 0.0001$. Then adjust w_{21} to reduce γ. We then get

$$W_2 = \begin{bmatrix} 0.12 & 0 \\ 0 & 0.0001 \end{bmatrix}$$

for which we get $\gamma = 3.3$ (5.4 dB and 35 deg).

The reference signal is injected at the outputs of the system. A prefilter

$$W_P = \begin{bmatrix} -0.001s/(0.25 + s) \\ 1 \end{bmatrix}$$

can be used to shape the response. The overshoot becomes about 20% and the rise time about 8 seconds.