1

Let (A, B, C, D) be a balanced realization of a system G. That is

$$A^*\Sigma + \Sigma A + C^*C = 0$$
$$A\Sigma + \Sigma A^* + BB^* = 0$$

(a,b) Is the balanced realization unique? If (A, B, C, D) is a balanced realization then (TAT^*, TB, CT^*, D) is also a balanced realization if $T^*T = I$ and $T\Sigma = \Sigma T$. Since, Σ is diagonal any $T = \operatorname{diag}[\pm 1, \pm 1, \ldots, \pm 1]$ will do.

(c) Consider the balanced realization of $G(s) = \frac{120-60s+12s^2-s^3}{120+60s+12s^2+s^3}$. Try to find a balanced realization such that as many elements as possible are zeroed in (A, B, C, D). In this case $\Sigma = I$. Then any unitary T will satisfy $T\Sigma = \Sigma T$.

```
sys = nd2sys ([-1 12 -60 120], [1 12 60 120]);
[bal, sig] = sysbal (sys);
P = sys2pss(bal); % turns the system in a [A B;C D] matrix
T1=daug ([orth(P(1:3,4)) null(P(1:3,4)')],1);
P1 = T1'*P*T1;
T2=daug (1,[orth(P1(2:3,1)) null(P1(2:3,1)')],1);
P2 = T2'*P1*T2
bal2 = pss2sys (P2, 3);
```

This yields the following system

$$\left[\begin{array}{ccc|c} -12 & -\sqrt{50} & 0 & \sqrt{24} \\ \sqrt{50} & 0 & -\sqrt{10} & 0 \\ 0 & \sqrt{10} & 0 & 0 \\ \hline \sqrt{24} & 0 & 0 & -1 \end{array}\right]$$

2

Let $G = \tilde{M}^{-1}\tilde{N}$ be a normalized coprime factorization. Perform a model reduction of $\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix}$ such that

$$\left\| \begin{bmatrix} (\tilde{N} - \tilde{N}_r) & (\tilde{M} - \tilde{M}_r) \end{bmatrix} \right\| \le \varepsilon.$$

Determine an upper bound for the relative error between $G_r = \tilde{M}_r^{-1}\tilde{N}_r$ and G

$$
\begin{aligned}
G - G_r &= \tilde{M}^{-1}\tilde{N} - \tilde{M}_r^{-1}\tilde{N}_r \\
&= \tilde{M}^{-1}(\tilde{N} - \tilde{N}_r) + (\tilde{M}^{-1} - \tilde{M}_r^{-1})\tilde{N}_r \\
&= \tilde{M}^{-1}(\tilde{N} - \tilde{N}_r) + \tilde{M}^{-1}(\tilde{M}_r - \tilde{M})\tilde{M}_r^{-1}\tilde{N}_r \\
&= \tilde{M}^{-1}\begin{bmatrix} (\tilde{N} - \tilde{N}_r) & (\tilde{M} - \tilde{M}_r) \end{bmatrix}\begin{bmatrix} I \\ G_r \end{bmatrix}
\end{aligned}
$$

Next we observe that $\tilde{M}\tilde{M}^{\sim} + \tilde{N}\tilde{N}^{\sim} = I$, or

$$
\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}^{\sim} = I
$$

Thus

$$
\tilde{M}^{-1} = \begin{bmatrix} I & \tilde{M}^{-1}\tilde{N} \end{bmatrix}\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}^{\sim} = \begin{bmatrix} I & G \end{bmatrix}\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}^{\sim}
$$

and

$$
\begin{aligned}
\|G - G_r\| &= \left\| \tilde{M}^{-1}\begin{bmatrix} (\tilde{N} - \tilde{N}_r) & (\tilde{M} - \tilde{M}_r) \end{bmatrix}\begin{bmatrix} I \\ G_r \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} I & G \end{bmatrix}\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}^{\sim}\begin{bmatrix} (\tilde{N} - \tilde{N}_r) & (\tilde{M} - \tilde{M}_r) \end{bmatrix}\begin{bmatrix} I \\ G_r \end{bmatrix} \right\| \\
&\leq \| \begin{bmatrix} I & G \end{bmatrix} \| \times 1 \times \varepsilon \times \left\| \begin{bmatrix} I \\ G_r \end{bmatrix} \right\| \\
&= \varepsilon \| \begin{bmatrix} I & G \end{bmatrix} \| \times \left\| \begin{bmatrix} I \\ G_r \end{bmatrix} \right\| \\
&= \varepsilon\sqrt{1 + \|G\|^2}\,\sqrt{1 + \|G_r\|^2}
\end{aligned}
$$

3

A transfer function G is called positive real if G is stable and $\operatorname{Re} G(j\omega) > 0$ for all $\omega \in \mathbb{R}$. Determine if $G(s) = D + C(sI - A)B$ is positive real without making a frequency sweep.

The system is positive real if $G(j\omega) + G(j\omega)^* > 0$ for all $\omega \in \mathbb{R}$. Specifically, we must require that $R = D + D^* > 0$. We further know that

$$
G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad G^{\sim} = \left[\begin{array}{c|c} -A^* & -C^* \\ \hline B^* & D^* \end{array}\right]
$$

Then

$$\Phi = G + G^{\sim} = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & -A^* & -C^* \\ \hline C & B^* & D + D^* \end{array}\right]$$

Note that $\Phi(j\omega)$ is real for all $\omega \in \mathbb{R}$. If G is positive real then $\Phi(j\omega) > 0$ for all $\omega \in \mathbb{R}$. Otherwise there exists at least on $\omega \in \mathbb{R}$ such that $\Phi(j\omega) = 0$. Thus Φ^{-1} has no poles on the imaginary axis if and only if G is positive real. The A-matrix of Φ^{-1}:

$$\begin{aligned} H &= \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C^* \end{bmatrix} R^{-1} \begin{bmatrix} C & B^* \end{bmatrix} \\ &= \begin{bmatrix} A - BR^{-1}C & -BR^{-1}B^* \\ C^*R^{-1}C & -(A - BR^{-1}C)^* \end{bmatrix} \end{aligned}$$

Thus G is positive real if and only if $D + D^* > 0$ and H has no imaginary eigenvalues.

Also refer to the ZDG book, section 13.5.

4

We will here consider the problem of controlling and stabilizing the attitude of a rocket using thrust vector control. A hypothetical rocket is used and we consider the control of the second stage, which has its burn phase in the upper atmosphere. The vehicle is aerodynamically unstable due to the fact that the center of pressure is in front of the center of mass. The vehicle is stabilized by directing its movable nozzle. The velocity of the vehicle is assumed to be relatively high, and we use the approximation that attitude is identical to angle of attack, which allows us to use only two states in the model. Neglecting aerodynamic damping we use

$$\ddot{y} = ay + bu$$

as our model of the vehicle dynamics, where y is the attitude of the vehicle, u is the thrust vector deflection, and a and b are parameters.

The parameters a and b are uncertain due to uncertainties in dynamic pressure (caused by altitude, velocity and air density), gravimetrics, and aerodynamics. In addition, the a parameter depends on the angle of attack.

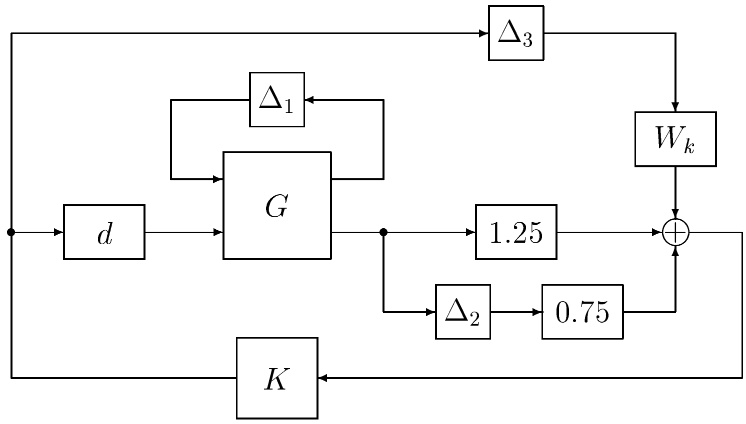

Figure 1: The augmented rocket plant

The parameter a can be modeled explicitly by an uncertainty while variations in b can be included in the phase and gain margin.

A delay of 0.06 seconds is included in the loop for modeling computational delay, sampling effects and actuator dynamics.

Design a controller that should satisfy the following requirements.

(i) $a = a_0 + a_1\Delta_1,\ \ \Delta_1 \in [0, 1]$;

(ii) The gain margin shall be 6 dB or better;

(iii) The phase margin shall be 35 degrees or better;

(iv) The compensator gain shall be less than –6 dB at frequencies above 50 rad/s.

A state-space model G is defined by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ a_0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_1 & b \end{bmatrix}\begin{bmatrix} w_1 \\ u \end{bmatrix} \\ \begin{bmatrix} z_1 \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{1}$$

where $a_0 = 0.5$, $a_1 = 0.5$ and $w_1 = \Delta_1 z_1$. The system is augmented according to figure 1 in order to take into account the requirements as uncertainties.

A delay of 0.06 seconds is included for modeling computational delay, sampling effects and actuator dynamics. This delay is implemented by a first order Padé approximation,

$$d(s) = \frac{1 - 0.03s}{1 + 0.03s}, \tag{2}$$

which is valid with a relatively good accuracy up to about 30 rad/s.

The gain and phase margins are assured by including a complex uncertainty in the feedback loop by the gain

$$\frac{1 + k_m \Delta_2}{\sqrt{1 - k_m^2}}, \tag{3}$$

with $k_m = 0.6$ and $|\Delta_2| < 1$.

For ensuring low enough gain at high frequencies the compensator gain is restricted by $|K(j\omega)| < |W_k(j\omega)|$ where

$$W_k(s) = 2\,\frac{(s + 7.436)^2 + 19.75^2}{(s + 7.07)^2 + 49.84^2}\,\frac{(s + 20.54)^2 + 9.36^2}{(s + 60.28)^2 + 72.89^2}. \tag{4}$$

This requirement comes from the fact that the rocket is not a rigid structure but has flexible modes due to the elasticity in structure and interstage joints. In a conservative design, like this one, we make no assumption on the phase since 50 rad/s is higher than the bandwidth of the system (determined by the total delay in the loop). Thus, stability is assured by the gain requirement for $\omega > 50$.

The system is augmented by the a-variation (Δ_1), the gain and phase margin requirement (Δ_2), and the high-frequency gain requirement (Δ_3) as depicted in figure 1.

μ Design

We start the design by D-K iterations. In the first iteration no scalings are used and the controller is obtained by a “straight” H_∞ synthesis. Next a frequency sweep of the closed loop system is performed and the maximum μ-value is computed with respect to frequency. The result is shown in table 1 on the first row.

In iteration 2, scalings are computed by fitting a state-space system to the frequency sweep data, using `musynfit` in the μ-Analysis and Synthesis Toolbox [?]. Third-order scalings are used for Δ_1 and Δ_2 while the scaling for Δ_3 is constant.

An abbreviated Matlab command sequence for the two first iterations is given below.

```
% iteration # 1: unscaled augmented system denoted by sysx
>> [k1, clp1] = hinfsyn (sysx, 1, 1, 2, 3, 0.0001);
  Gamma value achieved:     2.5715
% iteration #2
>> w = logspace (-1,2);
>> blk = [1 0; 1 0; 1 0];
>> [bnds, dvec, sens] = mu (frsp (clp, w), blk);
>> D = musynfit ('first', dvec1, sens1, blk, 1, 1);
>> mux = mmult (D, sysx, minv(D));
>> [k2, clp2] = hinfsyn (mux, 1, 1, 1, 1.2, 0.0001);
 Gamma value achieved:     1.1006
>> [bnds2, dvec2, sens2] = mu (frsp (clp2, w), blk);
```

We then repeat the H_∞ synthesis, this time on the scaled system. This time both the H_∞ norm and the μ-value are significantly lower than in the first iteration. Two more D-K-iterations are performed as is shown in table 1. The column denoted D+A scaling gives the orders of the scaling matrices D for Δ_1, Δ_2 and Δ_3 respectively. The scaling for Δ_3 is always one. The D-K-iterations converge after four iterations.

Table 1: Summary of the μ iterations. The order of the scaling matrices are given within brackets under the D-scaling column. For instance 3 denotes that D is of third order. The column denoted by H_∞ gives the H_∞ norm of the scaled system; μ shows the maximum value of μ with respect to frequency assuming that all uncertanties are dynamic.

iter	D scaling	H_∞	μ_C
1	-	2.5715	2.4620
2	[3, 3, 0]	1.1006	1.1003
3	[3, 3, 0]	1.0487	1.0487
4	[3, 5, 0]	1.0486	1.0487

We select the compensator from the third iteration since this is of slightly lower order than the fourth iteration compensator without any significant loss in performance. The third iteration compensator has 19 states and it is reduced to five states by first removing the fast dynamics (one pole) and replacing it by a constant and then performing a balanced realization on the normalized coprime factorization of the compensator. This gives a compensator of order five, which is almost indistinguishable from the original third iteration compensator. The μ value of the closed-loop system using the reduced compensator is 1.0605, which is slightly higher than the value obtained using the best compensator from the D-K algorithm.

$$K(s) = -0.466\,\frac{s+0.86}{s+26.24}\,\frac{(s+7.07)^2+49.85^2}{(s+5.97)^2+18.54^2}\,\frac{(s+60.24)^2+73.01^2}{(s+14.87)^2+16.04^2}. \tag{5}$$